HOMOLOGY OF COMPLEX PROJECTIVE HYPERSURFACES WITH ISOLATED SINGULARITIES

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ABSTRACT. We use information concerning the homology of links and Milnor fibers at each singularity of a hypersurface $\tilde{V}$, of degree d in CP^{n+1} with only isolated singularities, to determine $\tilde{V}$'s homology except for the question of torsion in $H_n(\tilde{V}; Z)$.

Consider a projective hypersurface $\tilde{V}$ of degree d in CP^{n+1} having only isolated singularities $P = \{p_1, \ldots, p_m\}$. Results of Howard [3] and Andreotti and Frankel [1] imply that we may regard $\tilde{V}$ as the closure of an affine hypersurface $V = f^{-1}(0) \subset C^{n+1}$ where $f: C^{n+1} \to C$ is a polynomial of degree d such that

(1) $P \subset V$,

(2) $\tilde{V}$ is nowhere tangent to the hyperplane at infinity:

$$H_0 = \{[z_0 : z_1 : \cdots : z_{n+1}] \in CP^{n+1} | z_0 = 0\}.$$

Given this situation we note that for sufficiently small $\varepsilon > 0$ the projective completion $\tilde{V}_\varepsilon$ of the affine hypersurface $V_\varepsilon = f^{-1}(\varepsilon)$ must be nonsingular. Furthermore $\tilde{V}_\varepsilon \cap H_0 = \tilde{V} \cap H_0$. By pushing along the vector field obtained by integrating $-\text{grad}_C f = \langle -\overline{\partial f}/\partial Z_1, \ldots, -\overline{\partial f}/\partial Z_{n+1} \rangle$ in C^{n+1} we obtain a map $\Phi: V_\varepsilon \to V$ and since $\tilde{V}$ is transverse to H_0 this extends to $\tilde{\phi}: \tilde{V}_\varepsilon \to \tilde{V}$ by defining $\tilde{\phi}|\tilde{V}_\varepsilon \cap H_0$ to be the identity. For $i = 1, \ldots, m$ and $\delta > 0$, let $S_{\delta,i}$ be a $(2n + 1)$-sphere of radius δ centered at $p_i \in C^{n+1}$, also $K_{\delta,i} = S_{\delta,i} \cap V$. Choose $\delta_1 > 0$ small enough so that for $\delta \leqslant \delta_1$, we get a Milnor fibration $F_i \to S_{\delta,i} \backslash K_{\delta,i} \xrightarrow{\theta_i} S^1$. Here $\theta_i(Z) = f(z)/|f(z)|$. Let $\delta_0 < \delta_1$ and $D_{i,0}$ ($D_{i,1}$) be the open $2n$-disk with radius δ_0 (δ_1), respectively, around p_i.

THEOREM 1. *Letting* $D_1 = \bigcup_{i=1}^m D_{i,1}$ *and* $D_0 = \bigcup_{i=1}^m D_{i,0}$, $\tilde{\phi}|(\tilde{V}_\varepsilon \backslash D_0)$ *is isotopic to a diffeomorphism* $(\tilde{V}_\varepsilon \backslash D_0) \approx (\tilde{V} \backslash D_0)$ *for sufficiently small* $\varepsilon > 0$.

PROOF. Choose ε small enough so that

(1) $V_\varepsilon \cap D_{i,0} \approx F_i$ the Milnor fiber at p_i; see [6, Lemma 5.11].

(2) $f^{-1}(\gamma)$ is nonsingular for $0 < |\gamma| < \varepsilon$.

(3) $\phi^{-1}[p_i] \subset V_\varepsilon \cap D_{i,0}$ thus insuring that ϕ is 1-1 on $V_\varepsilon \backslash D_0$.

The fact that $f^{-1}(\gamma)$ is transverse to $S_{\delta,i}$ for $0 < |\gamma| \leqslant \varepsilon$ and $\delta_0 \leqslant \delta \leqslant \delta_1$

means that the component of $-\mathrm{grad}_C f$ which is tangent to S_δ is nonvanishing. Hence we may obtain a new vector field by using a partition of unity in the collar $D_1 \backslash D_0$ to smoothly patch $-\mathrm{grad}_C f$ in $f^{-1}[0, \varepsilon] \backslash D_1$ with the tangential component of $-\mathrm{grad}_C f$ on X_0. Pushing along this vector field yields the diffeomorphism $V_\varepsilon \backslash D_0 \approx V \backslash D_0$. By smoothly altering the partition of unity, one shows this diffeomorphism to be isotopic to $\phi \colon V_\varepsilon \backslash D_0 \to V \backslash P$. Taking projective completions and defining the levels of the isotopy to be the identity on $\tilde{V}_\varepsilon \cap H_0$ completes the theorem.

COROLLARY 2. *Letting W denote a nonsingular hypersurface of degree d in CP^{n+1} we have the following cohomology ladder (Z-coefficients).*

$$\cdots \longrightarrow H^{q-1}(K) \xrightarrow{\delta} H^q(\tilde{V}) \xrightarrow{j} H^q(\tilde{V} \backslash D) \oplus H^q(P) \longrightarrow H^q(K) \longrightarrow \cdots$$

$$\approx \Big\downarrow \qquad \psi_q \Big\downarrow \qquad \psi_q | \oplus \tau \Big\downarrow \qquad \approx \Big\downarrow$$

$$\cdots \longrightarrow H^{q-1}(K) \longrightarrow H^q(W) \xrightarrow{j} H^q(W \backslash D) \oplus H^q(F) \longrightarrow H^q(K) \longrightarrow \cdots$$

Here F is the disjoint union of the Milnor fibers at each point of the singular locus $P = \{p_1, \ldots, p_m\}$, $K = \delta F$, and $\psi_q | \colon H^q(\tilde{V} \backslash D) \to H^q(W \backslash D)$ is an isomorphism for all q.

PROOF. Consider the Mayer-Vietoris sequence resulting from the following adjunctions:
  (i) $\tilde{V} = (\tilde{V} \backslash D_0) \cup_{K_0}$ Cones on $K_{i,0}$;
  (ii) $W = \tilde{V}_\varepsilon \approx (\tilde{V}_\varepsilon \backslash D_0) \cup_{K_0} \bar{F}$, where $\bar{F} = \cup \bar{F}_i$ and $\bar{F}_i = D_0 \cap \tilde{V}_\varepsilon$ and is diffeomorphic to the closed Milnor fiber at p_i.
  The vertical maps are induced by $\tilde{\phi}$ and since $H^q(P) = 0$ for $q \geqslant 1$, τ is the zero map; for $q = 0$, τ is an isomorphism. $\tilde{\psi}_q |$ is an isomorphism by Theorem 1.

THEOREM 3. *If $\tilde{V}$ is a projective hypersurface $\subset CP^{n+1}$ of degree d with singular locus $P = \{p_1, \ldots, p_m\}$, then*
  (i)

$$H^q(\tilde{V}; Z) = \begin{cases} Z, & q \text{ even} \neq n, n+1, \\ 0, & q \text{ odd} \neq n, n+1, \end{cases} \qquad q \leqslant 2n;$$

(ii) *Free $H^{n+1}(\tilde{V}; Z)$ has rank*

$$\begin{cases} 1 + \sum_1^m \tilde{\beta}_{n-1}(K_i), & n \text{ odd}, \\ \sum_1^m \tilde{\beta}_{n-1}(K_i), & n \text{ even}; \end{cases}$$

(iii) *$H^n(\tilde{V}; Z)$ is free with rank $\beta_n(W) - \sum_1^m \mu_i + \sum_1^m \tilde{\beta}_{n-1}(K_i)$ where $\beta_n(W) = \mathrm{rank}\, H^n(W)$, and is well known to equal*

$$\begin{cases} d^{-1}(d-1)[(d-1)^{n+1} - 1], & n \text{ odd}, \\ d^{-1}[(d-1)^{n+2} - 1] + 2, & n \text{ even}, \end{cases}$$

and $\mu_i = H^n(F_i)$. $\tilde{\beta}_{n-1}(K_i)$ denotes the reduced rank of $H^{n-1}(K_i)$.

(iv) $\delta|$: Tor $H^n(K;Z) \to$ Tor $H^{n+1}(\tilde{V};Z)$ *is epic.*

PROOF. Statements (i), (ii), and (iii) follow from the facts that K is an $(n-1)$-connected $(2n-1)$-manifold, F is a $2n$-manifold with $\partial F = K$ and is of the same homotopy type as a wedge of S^ms, and the cohomology ladder of Corollary 2. Suppose $x \in H^n(\tilde{V};Z)$ and $mx = 0$. Hence $m \cdot \psi_n(x) = 0$ and, since $H^n(W;Z)$ is free, $\psi_n(x) = 0$. Thus $j \cdot \psi_n(x) = (\psi_n|\oplus \tau) \cdot j(x) = 0$. But $H^n(P;Z) = 0$ and $\psi_n|$ is an isomorphism. Hence $j(x) = 0$ and $\exists y \in H^{q-1}(K)$ such that $\delta(y) = x$ by exactness.

BIBLIOGRAPHY

1. A. Andreotti and T. Frankel, *The second Lefschetz theorem on hyperplane sections*, Global Analysis (Papers in Honor of K. Kodaira), Univ. of Tokyo Press, Tokyo, 1969, pp. 1–20. MR **42** #5989.

2. F. Hirzebruch, W. Neumann and S. Koh, *Differentiable manifolds and quadratic forms*, Dekker, New York, 1971. MR **49** #6250.

3. A. Howard, *On the homotopy groups of an affine algebraic hypersurface*, Ann. of Math. (2) **84** (1966), 197–216. MR **35** #4946.

4. M. Kato and Y. Matsumoto, *Simply connected surgery of submanifolds in codimension two*. I (preprint).

5. J. Milnor, *Morse theory*, Ann. of Math. Studies, no. 51, Princeton Univ. Press, Princeton, N. J., 1963. MR **29** #634.

6. ———, *Singular points of complex hypersurfaces*, Ann. of Math. Studies, no. 61, Princeton Univ. Press, Princeton, N. J.; Univ. of Tokyo Press, Tokyo, 1968. MR **39** #969.

7. J. Milnor and P. Orlik, *Isolated singularities defined by weighted homogeneous polynomials*, Topology **9** (1970), 385–393. MR **45** #2757.

8. E. Thomas and J. Wood, *On manifolds of representing homology classes in codimension 2*, Invent. Math. **25** (1974), 63–89.

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