ON THE STRUCTURE OF LINDENBAUM ALGEBRAS:
AN APPROACH USING ALGEBRAIC LOGIC

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ABSTRACT. The following problem of algebraic logic is investigated: to
determine those Boolean algebras which admit the structure of a nondiscrete
cylindric algebra. A partial solution is found, and is then used to give an
algebraic characterization of the Lindenbaum algebras of formulas of several
broad classes of countable theories.

1. Introduction. A major open problem of algebraic logic is the following:
Which Boolean algebras admit the structure of a nondiscrete cylindric or
polyadic algebra? Using results of Henkin, Monk and Tarski [1], one easily
proves:
A denumerable Boolean algebra admits the structure of a nondiscrete, dimen-
sion-complemented cylindric algebra if and only if it is not atomic.

We establish this, as well as a few related results, and use them to investigate
the structure of Lindenbaum algebras of countable theories.

In the sequel, let $T$ denote any countable theory. By the Lindenbaum
algebra $\mathcal{F}_T$ of $T$ we will always mean the Lindenbaum algebra of formulas\(^1\) of
$T$. From results in [1] we easily establish that if $T$ has no one-element models
then $\mathcal{F}_T$ is atomless (this characterizes $\mathcal{F}_T$, for there is, up to isomorphism, only
one atomless, denumerable Boolean algebra). Let us say that $T$ admits
elimination of all but $n$ predicates if $T$ is definitionally equivalent to a theory $T'$
whose language may have finitely or denumerably many operation symbols, but
has no more than $n$ predicate symbols other than $=$; for $n = 0$, we say that
$T$ admits elimination of predicates. It is shown that an arbitrary theory $T$ admits
elimination of predicates if and only if $T$ has either no one-element models, or
all of its one-element models are elementarily equivalent. We prove that if a
theory $T$ admits elimination of predicates then $\mathcal{F}_T$ has $\leq 1$ atom; more
generally, if $T$ admits elimination of all but $n$ predicates then $\mathcal{F}_T$ has $\leq 2^n$
atoms. Then we provide a method to determine the exact structure of $\mathcal{F}_T$
whenever $T$ admits elimination of all but finitely many predicates.

Our notation and terminology is that of Henkin, Monk and Tarski [1], and
we presuppose an acquaintance at least with Chapter 1 of this work.

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\(^1\) We will deal here with Lindenbaum algebras of formulas, rather than Lindenbaum algebras
of sentences.
2. Results on cylindric algebras. Throughout this section, let \( \mathcal{A} = \langle A, +, \cdot, -, 0, 1, c_\kappa, d_\kappa \rangle_{\kappa, \lambda < \alpha} \) be a nondiscrete, dimension-complemented cylindric algebra. By [1, 1.11.3(iii)], \( \alpha \geq \omega \). The following statements, which are easily deduced from results given in [1], will be needed in the sequel:

(A) if \( c_0^0 d_{01} = 0 \), then \( \mathcal{A} \) is atomless;

(B) if \( c_0^0 d_{01} \neq 0 \), and there is no zero-dimensional element \( x \neq 0 \) such that \( x < c_0^0 d_{01} \), then \( c_0^0 d_{01} \) is the only atom of \( \mathcal{A} \);

(C) \( \mathcal{A} \cong \mathcal{B} \times \mathcal{C} \) where \( \mathcal{B} \) is atomless and \( \mathcal{C} \) is discrete.

(A) and (B) follow from [1, Theorems 1.10.5(a), 1.11.8(i) and 1.6.203]. (C) follows from [1, Theorems 2.4.37 and 1.11.8(ii)].

Every Boolean algebra admits the structure of a discrete cylindric algebra, so there is no need to consider that case further. Similarly, every Boolean algebra admits the structure of a cylindric algebra of degree 1, for example by taking \( c \) to be the quantifier given by \( c0 = 0, x \neq 0 \Rightarrow cx = 1 \). Thus, we should confine our attention to nondiscrete cylindric algebras of degree \( \alpha \geq 2 \).

If \( \mathcal{B} \) is any denumerable, atomless Boolean algebra, then \( \mathcal{B} \) admits the structure of a nondiscrete cylindric algebra of degree \( \omega \). Indeed, if \( T \) is any countable theory which has no one-element models, then \( T \vdash \neg(\forall v_0) (v_0 = v_1) \), hence by (A), the Lindenbaum algebra of formulas of \( T \) is an atomless denumerable Boolean algebra. This Lindenbaum algebra is isomorphic to \( \mathcal{B} \) because any two denumerable atomless Boolean algebras are isomorphic.

Now, let \( \mathcal{A} \) be any denumerable Boolean algebra having a direct factor which is an atomless denumerable Boolean algebra, say \( \mathcal{A} \cong \mathcal{B} \times \mathcal{C} \) where \( \mathcal{B} \) is atomless and denumerable. We have just seen that \( \mathcal{B} \) admits the structure of a nondiscrete, dimension-complemented cylindric algebra, and \( \mathcal{C} \) certainly admits the structure of a discrete cylindric algebra, hence \( \mathcal{A} \) admits the structure of a nondiscrete, dimension-complemented cylindric algebra. Combining this with (C), we get

(D) A denumerable Boolean algebra admits the structure of a nondiscrete dimension-complemented cylindric algebra if and only if it has a direct factor which is denumerable and atomless.

By the elementary theory of Boolean algebras, to say that \( \mathcal{A} \) has a direct factor which is denumerable and atomless is equivalent to saying that \( \mathcal{A} \) is not atomic. Thus, we have proved

**Theorem 1.** A denumerable Boolean algebra admits the structure of a nondiscrete dimension-complemented cylindric algebra if and only if it is not atomic.

In the discussion which follows we will use an algebraic counterpart of terms in first-order languages. For a full discussion of terms in cylindric algebras the reader is referred to [4]; however, for the present purposes only a few rudimentary notions are needed. An element \( x \in A \) will be called "diagonal-like" if it has the following two properties for some \( \kappa < \alpha \):

1. \( c_\kappa x = 1 \), and
2. \( x \cdot s_\mu x \leq d_{\kappa \mu} \) for each \( \mu \in \alpha - \Delta x \).
With every diagonal-like element \( x \in A \) we associate a term \( a \), and (for \( x \) satisfying (1) and (2) above), we write \( x = d_{ka} \). (In the metalogical interpretation, \( d_{ka} \) is the equivalence class of the formula \( v_k = a \), and (1) and (2) assert the unique existence of \( v_k \) satisfying \( v_k = a \). Thus, if \( \mathcal{A} \) is taken to be an algebra of formulas, the "terms" of \( \mathcal{A} \) are all the terms which are explicitly definable in the theory associated with \( \mathcal{A} \).)

The following properties of diagonal-like elements will be relevant to our discussion:

**Theorem 2.** If \( x \) is any diagonal-like element, then \( x \geq c^0_\partial d_{01} \).

**Proof.** Let \( x \) satisfy (1) and (2). By [1, 1.6.20], \( c^3_k d_{k\lambda} - x \in Zd\mathcal{A} \). Thus,
\[
c^3_k d_{k\lambda} \cdot -x = c^3_k (c^3_k d_{k\lambda} \cdot -x) = c^3_k d_{k\lambda} \cdot -c_k x = c^3_k d_{k\lambda} \cdot 0 = 0.
\]
Thus, \( c^3_k d_{k\lambda} \leq x \).

From this theorem, we deduce a useful generalization of [1, Theorem 2.3.33]:

**Corollary 3.** Suppose \( \mathcal{A} \in Da_\alpha, \alpha \geq 2, \) and \( \mathcal{A} \) has a set of generators, \( X \), such that all but \( n \) elements of \( X \) are diagonal-like. Then

(i) \( |At\mathcal{A}| \leq 2^n \), and
(ii) \( c^0_\partial d_{01} = \sum At\mathcal{A} \).

**Proof.** By Theorem 2, if \( x \) is diagonal-like, then \( x \cdot c^0_\partial d_{01} = c^0_\partial d_{01} \). The remainder of the argument is exactly as in [1, Theorems 2.3.31 and 2.3.33].

The converse of Theorem 2, which follows next, states that if \( x \geq c^0_\partial d_{01} \), then \( x \) is generated from the diagonal-like elements of \( A \).

**Theorem 4.** If \( x \geq c^0_\partial d_{01} \), then there is a diagonal-like element \( y \) such that \( x = -c_k c_\lambda c_\mu (y \cdot d_{k\mu} - d_{k\lambda}) \).

**Proof.** Take distinct \( \kappa, \lambda, \mu \in \alpha - \Delta x \). Let
\[
y = d_{k\lambda} \cdot d_{k\mu} + d_{k\mu} - d_{k\lambda} - x + d_{k\mu} - d_{k\lambda} \cdot x.
\]
One verifies directly (we omit the simple details) that \( c_\mu y = 1 \), and for any \( \nu \in \alpha - \Delta y, y \cdot s^\mu_\nu y \leq d_{\mu\nu} \). Thus, \( y \) is a diagonal-like element. We note that
\[
c_\kappa c_\lambda c_\mu (-d_{k\lambda} \cdot y \cdot d_{k\mu}) = c_\kappa c_\lambda [-d_{k\lambda} \cdot c_\mu (y \cdot d_{k\mu})] = c_\kappa c_\lambda (-d_{k\lambda} \cdot s^\mu_\nu y).
\]
Now, \( s^\mu_\nu y = d_{k\lambda} + -d_{k\lambda} \cdot -x \), hence
\[
c_\kappa c_\lambda (-d_{k\lambda} \cdot s^\mu_\nu y) = c_\kappa c_\lambda (-d_{k\lambda} \cdot [d_{k\lambda} + -d_{k\lambda} \cdot -x])
= c_\kappa c_\lambda (-d_{k\lambda} \cdot -x) = (c_\kappa c_\lambda - d_{k\lambda}) \cdot -x.
\]
But by assumption, \( -x \leq -c^0_\partial d_{01} = c_\kappa c_\lambda - d_{k\lambda}, \) so \( c_\kappa c_\lambda (-d_{k\lambda} \cdot s^\mu_\nu y) = -x \).

**Corollary 5.** \( \mathcal{A} \) is generated by its diagonal-like elements iff \( c^0_\partial d_{01} = 0 \) or \( c^3_k d_{01} \) is an atom.

**Proof.** If \( X \) is a set of generators of \( \mathcal{A} \), then (as in the proof of [1, 2.3.31],
\( R_{c_0^d d_0^1} \mathcal{A} \) is generated by \( \{ x \cdot c_0^d d_0^1 : x \in X \} \). Thus, if \( X \) contains only diagonal-like elements, then by Theorem 2, \( \{ x \cdot c_0^d d_0^1 : x \in X \} = \{ c_0^d d_0^1 \} \), hence \( c_0^d d_0^1 = 0 \) or \( c_0^d d_0^1 \) is an atom. Conversely, suppose that \( c_0^d d_0^1 = 0 \) or \( c_0^d d_0^1 \) is an atom. In the first case, \( x \geq c_0^d d_0^1 \) for every \( x \in A \); in the second case, either \( x \geq c_0^d d_0^1 \) or \( -x \geq c_0^d d_0^1 \) for every \( x \in A \). Thus, by Theorem 4, \( \mathcal{A} \) is generated by its diagonal-like elements.

Finally, the following result is of some interest:

**Theorem 6.** \( \mathcal{A} \) has a set of generators, \( X \), which contains only diagonal-like elements and zero-dimensional elements.

**Proof.** \( \mathcal{A} \cong R_{c_0^d d_0^1} \mathcal{A} \times R_{-c_0^d d_0^1} \mathcal{A} \); as we have seen above, \( R_{-c_0^d d_0^1} \mathcal{A} \) is generated by its diagonal-like elements, and \( R_{c_0^d d_0^1} \mathcal{A} \) is generated by zero-dimensional elements.

### 3. Applications to Lindenbaum algebras.

The connections between algebraic logic and logic are studied in [2] and [3]. For example, it is proved in [3] that two arbitrary theories are definitionally equivalent iff their associated cylindric algebras are isomorphic. Furthermore, from the discussion in [3], it is clear that if a language has no relation symbols, then its associated cylindric algebra is generated by its diagonal-like elements; and conversely, if the cylindric algebra associated with a theory \( T \) is generated by its diagonal-like elements, then \( T \) is definitionally equivalent to a theory in a language with no relation symbols. In the sequel, these facts will be used without further explicit mention.

Throughout this section, we will take \( \mathcal{A} \) to be the Lindenbaum algebra of formulas, \( \mathcal{S}_T \), of a first-order theory \( T \).

We will show, first, that Corollary 5 yields a necessary and sufficient condition for the eliminability of predicates in favor of functions. We begin by noting the following:

For \( \mathcal{A} = \mathcal{S}_T \), \( c_0^d d_0^1 = 0 \) iff \( T \vdash \neg (\forall \nu) \) \((\nu_0 = \nu_1) \) iff \( T \) has no one-element models. On the other hand, \( c_0^d d_0^1 \) is an atom if and only if \( R_{c_0^d d_0^1} \mathcal{A} \) is a two-element Boolean algebra. Now, the discrete cylindric algebra \( R_{c_0^d d_0^1} \mathcal{A} \) is the algebra of formulas of the theory of one-element models of \( T \), and a theory is complete iff its Boolean algebra of sentences has two elements; thus, \( c_0^d d_0^1 \) is an atom iff all the one-element models of \( T \) are elementarily equivalent. Combining this with Corollary 5, we get

(E) An arbitrary theory \( T \) admits elimination of predicates iff \( T \) either has no one-element models, or all of its one-element models are elementarily equivalent.

From Theorem 6 we deduce that for any theory \( T \), predicate symbols are eliminable in favor of function symbols and propositional constants:

(F) Any theory \( T \) is definitionally equivalent to a theory \( T' \) whose language has no predicate symbols but may have function symbols and propositional constants.

(Note that the propositional constants serve only to axiomatize the class of one-element models of \( T \).)

We will now use Corollary 3 to describe the structure of the Lindenbaum algebras of formulas of certain theories. If \( T \) has no one-element models, then, as we have already noted, in \( \mathcal{S}_T \), \( c_0^d d_0^1 = 0 \). Thus, in view of (A), we have
(G) If $T$ has no one-element models, then $\mathcal{F}_T$ is an atomless Boolean algebra.

If $T$ has one-element models, we may use Corollary 3 to deduce

(H) If $T$ admits elimination of all but $n$ predicates, then $\mathcal{F}_T$ has $\leq 2^n$ atoms.

If $T$ is any theory which admits elimination of all but finitely many predicates we can, in fact, find the exact structure of $\mathcal{F}_T$. We assume the language $L$ of $T$ is denumerable.

Let $\langle P_i \rangle_{1 \leq i \leq n}$ be the sequence of predicate symbols of $L$, and let $\delta_i$ be the rank of $P_i$ for each $i \leq n$. It follows from [1, Theorem 2.4.37] that $\mathcal{F}_T \cong \mathfrak{B} \times \mathfrak{C}$, where $\mathfrak{B} = Rl_{c,\delta_0} \mathfrak{A}$ and $\mathfrak{C} = Rl_{c,\delta_0} \mathfrak{A}$. We have already seen that $\mathfrak{C}$ belongs to the isomorphism class of denumerable, atomless Boolean algebras, so it remains only to determine the structure of $\mathfrak{B}$. Now, $\mathfrak{B}$ is the algebra of formulas of the theory $T_1$ whose nonlogical axioms are those of $T$ together with the formula $(\forall v_0) (v_0 = v_1)$. It is immediately verified that

$$T_1 \vdash P_i(v_1, \ldots, v_{k_i}) \iff P_i(t_1, \ldots, t_{k_i})$$

for every $i \leq n$ and all terms $t_j$, and $T_1 \vdash (\forall v_k) F \iff F$ for every formula $F$. Thus $\mathfrak{B}$, the algebra of formulas of $T_1$, is the same as the algebra of formulas of the theory in the propositional calculus whose propositional variables are $P_1, \ldots, P_{n-1}$, and whose axioms are obtained from those of $T_1$ by deleting all variables, terms and quantifiers (together with the associated commas and brackets).

**BIBLIOGRAPHY**


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