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THE PRIME RADICAL IN ALTERNATIVE RINGS

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ABSTRACT. The characterization by J. Levitzki of the prime radical of an associative ring R as the set of strongly nilpotent elements of R is adapted here to apply to a wide class of nonassociative rings. As a consequence it is shown that the prime radical is a hereditary radical for the class of alternative rings and that the prime radical of an alternative ring coincides with the prime radical of its attached Jordan ring.

In 1951 J. Levitzki characterized the prime radical of an associative ring R as the set of all elements $r \in R$ such that every m-sequence beginning with r ends in zero [3]. An m-sequence was defined to be a sequence $\{a_0, a_1, \ldots, a_n, \ldots\}$ such that $a_i \in a_{i-1} R a_{i-1}$ for $i = 1, 2, \ldots$. Recently, C. Tsai [9] has given a similar characterization for the prime radical of a Jordan ring. Here we extend this characterization to the class of all s-rings and, as a consequence, are able to show that the prime radical is a hereditary radical on the class of alternative rings (i.e., if A is an ideal of an alternative ring R, then $P(A) = A \cap P(R)$) and that $P(R) = P(R^+)$ for all 2 and 3-torsion free alternative rings R. Although it is not known whether the prime radical is hereditary on the class of all s-rings, a partial result in this direction is obtained.

Recall that a not necessarily associative ring R is called an s-ring for a positive integer s if A^s is an ideal of R whenever A is an ideal of R (A^s denotes the set of all sums of products $a_1 a_2 \cdots a_s$ for $a_i \in A$ under all possible associations). An ideal P of R is called a prime ideal if whenever $A_1 A_2 \cdots A_s \subseteq P$ then $A_i \subseteq P$ for some i. Here $A_1 A_2 \cdots A_s$ denotes the product of the ideals under all possible associations. The prime radical, $P(R)$, of R is the intersection of all prime ideals of R and can be characterized as the set of all elements $r \in R$ such that every complementary system M of R which contains r also contains 0. A set M in R is a complementary system if whenever $A_1, A_2, \ldots, A_s$ are s ideals of R such that $A_i \cap M \neq \varnothing$ for $i = 1, 2, \ldots, s$, then $(A_1 A_2 \cdots A_s) \cap M \neq \varnothing$ [6], [8], [10].

To make this article self-contained we mention the following three properties of $P(R)$ which hold for any s-ring R. Proofs can be found in [6], [8] and [10].

(a) $P(R) = 0$ if and only if R contains no nonzero nilpotent ideals.

(b) $P(R/P(R)) = 0$.

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(c) $P(R)$ is the intersection of all ideals Q of R such that R/Q contains no nonzero nilpotent ideals.

In what follows (a) denotes the principal ideal generated by a.

DEFINITION. A sequence $\{a_0, a_1, \ldots, a_n, \ldots\}$ in an s-ring R is called a P-sequence if $a_n \in (a_{n-1})^s$ for $n = 1, 2, \ldots$. An element a of R is called strongly nilpotent if every P-sequence beginning with a ends in zero.

THEOREM 1. *In any s-ring R, the prime radical P(R) consists of all strongly nilpotent elements of R.*

PROOF. Suppose that $a \in P(R)$ and that $M = \{a_0 = a, a_1, \ldots, a_n, \ldots\}$ is a P-sequence beginning with a. Let $A_1, A_2, \ldots, A_s$ be ideals of R such that $A_i \cap M \neq \varnothing$ for each i. Thus, there exist elements $a_{i_1} \in A_1 \cap M$, $a_{i_2} \in A_2 \cap M, \ldots, a_{i_s} \in A_s \cap M$. Since by hypothesis $a_i \in (a_{i-1})^s$, it follows that if $i \leqslant j$ then $(a_j) \subseteq (a_i)$. Let $t = \max(i_1, i_2, \ldots, i_s)$. Then $a_{t+1} \in (a_t)^s \subseteq (a_{i_1})(a_{i_2}) \cdots (a_{i_s}) \subseteq A_1 A_2 \cdots A_s$. We have shown that if $A_i \cap M \neq \varnothing$ for all i then $(A_1 A_2 \cdots A_s) \cap M \neq \varnothing$. Thus M is a complementary system. Since $a \in M \cap P(R)$ it follows that $0 \in M$. Thus $a_l = 0$ for some l and a is a strongly nilpotent element.

Conversely, if $a \notin P(R)$ then there is a prime ideal P such that $a \notin P$. Let S be the sequence $\{a_0, a_1, \ldots, a_n, \ldots\}$ where $a_0 = a$, $a_1 \in (a_0)^s \cap C(P)$, $\ldots, a_n \in (a_{n-1})^s \cap C(P), \ldots$. Such a sequence can be defined inductively since if a_{n-1} has been chosen in $C(P)$ then $(a_{n-1})^s \not\subset P$, so that there exists $a_n \in (a_{n-1})^s \cap C(P)$. Thus, S is a P-sequence beginning in a which does not end in zero. It follows that a is not strongly nilpotent. We conclude that $P(R)$ is precisely the set of strongly nilpotent elements of R.

It is well known that the prime radical is hereditary on associative rings; i.e., if R is an associative ring and if A is an ideal of R then $P(A) = A \cap P(R)$ [1]. We proceed to investigate this concept for s-rings in general.

DEFINITION. A ring R is called a strongly s-ring if every subring of R is an s-ring.

Clearly every variety all of whose members are s-rings is a variety of strongly s-rings. Thus, the varieties of associative, alternative and Lie rings are strongly 2-varieties and the varieties of Jordan and standard rings are strongly 3-varieties [6].

LEMMA. *If R is a strongly s-ring and if B is a subring of R then $B \cap P(R) \subseteq P(B)$.*

PROOF. Since B is an s-ring, $P(B)$ is the set of strongly nilpotent elements in B. Let $b \in B \cap P(R)$. Then any P-sequence of R beginning with b ends in zero. Let S be a P-sequence of B beginning with b. Then S is clearly a P-sequence of R beginning with b. Thus S ends in zero so that $b \in P(B)$.

THEOREM 2. *The prime radical is hereditary on alternative rings R (i.e., if A is an ideal of R then $P(A) = A \cap P(R)$).*

PROOF. Since R is a strongly 2-ring it follows by the Lemma that $A \cap P(R) \subseteq P(A)$. Now Slater [7, Theorem C] has shown that if R contains no nilpotent ideals then A (as a ring) contains no nilpotent ideals. This translates to "$P(R) = 0$ implies $P(A) = 0$" and is sufficient to prove the converse. For, suppose that $P(R) \neq 0$. Then, by (b), $P(R/P(R)) = 0$. Thus, by Slater's result it follows that $P((A + P(R))/P(R) \cong P(A/(A \cap P(R))) = 0$. Thus

$$A/(A \cap P(R))$$

contains no nonzero nilpotent ideals. Since $P(R)$ can be characterized as the intersection of all ideals Q of R such that R/Q contains no nonzero nilpotent ideals [6], it follows immediately that $P(A) \subseteq A \cap P(R)$.

REMARKS. 1. A property P of rings is called a radical property in the sense of Kurosh if: (a) homomorphic images of P-rings are P-rings, (b) every ring R contains a P-ideal S which contains every P-ideal, and (c) R/S contains no nonzero P-ideals. It is known that the prime radical is a radical property for associative rings [1, p. 57]. In view of the fact that the prime radical coincides with the Baer lower radical in any s-ring [6], and the fact that in any alternative ring R, $P(R) = 0$ implies $P(A) = 0$ for all ideals A of R, we may observe that the proof in [1] that the prime radical is a radical property for associative rings can be adopted word for word to show that the prime radical is a radical property for alternative rings. Note also, by virtue of the proof of Theorem 2, that in any strongly s-ring in which $P(R) = 0$ implies $P(A) = 0$ for all ideals A of R, the prime radical is hereditary. I do not know if this is the case for Jordan and Lie rings.

2. In view of the fact that in an s-ring R prime ideals are defined in terms of the number s, it is necessary to show that if R is an s-ring and also a t-ring for $s \neq t$, that $P(R)$ is unchanged whether R is viewed as an s-ring or as a t-ring. However, the characterization of $P(R)$ used in the proof of Theorem 2 as the intersection of all ideals Q of R such that R/Q contains no nonzero nilpotent ideals does the trick, since nilpotent ideals are not defined in terms of the number s.

If R is an alternative ring the attached ring R^+ of R is setwise equal to R but multiplication in R^+ is given by $a \cdot b = ab + ba$ where ab is the multiplication in R. It is well known that if R is alternative then R^+ is a special Jordan ring. Thus it is of interest to relate the radical properties of R to those of R^+. Using the structure theory for simple alternative rings, it is easy to see that R is simple if and only if R^+ is simple. McCrimmon [5] has shown that $J(R) = J(R^+)$ where J denotes the Jacobson radical. The following theorem treats the same problem for the prime radical and generalizes the result of Erickson and Montgomery [2] who obtained the result for associative rings. We shall say that R is n-torsion free if $nr = 0$ for $r \in R$ implies $r = 0$.

THEOREM 3. *If R is a 2 and 3-torsion free alternative ring then* $P(R) = P(R^+)$.

PROOF. Since R^+ is a Jordan ring it is a strongly 3-ring. By $[a]$ we denote the principal ideal generated by a in R^+ and by $[a]^n$ we always mean multiplication in R^+. Let $a \in P(R)$. Then any sequence $\{a, a_1, \ldots, a_n, \ldots\}$ where $a_1 \in (a)^2$ and $a_i \in (a_{i-1})^2$ for all i ends in zero. Let $S = \{a = b_0, b_1, \ldots, b_n, \ldots\}$ be a P-sequence of R^+ beginning in a. Thus, $b_i \in [b_{i-1}]^3$ for all i. However $[b_{i-1}]^3 \subseteq [b_{i-1}]^2 \subseteq (b_{i-1})^2$ for all i. Thus S is a P-sequence of R beginning with a, from which it follows that $0 \in S$. Hence a is a strongly nilpotent element of R^+ and by Theorem 1, $a \in P(R^+)$. Thus, $P(R) \subseteq P(R^+)$.

For the converse it is sufficient to show that $P(R) = 0$ implies $P(R^+) = 0$. For, under this assumption, if $P(R) \neq 0$ then we consider $P(R/P(R))$. Since $P(R/P(R)) = 0$ we get $P((R/P(R))^+) = 0$. But $(R/P(R))^+ \cong R^+/P(R)^+ = R^+/P(R)$. Thus $P(R^+/P(R)) = 0$ from which it follows that $R^+/P(R)$ contains no nonzero nilpotent ideals. But $P(R^+)$ is the intersection of all ideals Q of R^+ such that R^+/Q contains no nonzero nilpotent ideals. Thus $P(R^+) \subseteq P(R)$.

It remains to show that $P(R) = 0$ implies that $P(R^+) = 0$. But if $P(R^+) \neq 0$ then there exists a nonzero nilpotent ideal of R^+. Since R^+ is a 3-ring, we can assume without loss of generality that there exists a nonzero ideal U of R^+ such that $(U^+)^3 = 0$ ($(U^+)^3$ denotes the 3rd-power of U in the ring R^+). Let $u \in U$, $r \in R$. Then $v = u^2 \cdot r \in U$ and since $(U^+)^3 = 0$ it follows that $u^2 \cdot v = 0$. But

$$u^2 \cdot v = u^2(u^2r + ru^2) + (u^2r + ru^2)u^2 = u^4r + 2u^2ru^2 + ru^4$$

since R is alternative. But $u^4 = 0$ and R is 2-torsion free. Thus, we have $u^4 = u^2ru^2 = 0$ for all $r \in R$. Since R is 3-torsion free and $P(R) = 0$, a result of Kleinfeld [4, Lemma 1] yields $u^2 = 0$ for all $u \in U$. Therefore $w \cdot u = 0$ for all $w, u \in U$. Let $u \in U$ and $w = u \cdot r$ for $r \in R$. Then $w \in U$ so that $u \cdot w = 0$. But $u \cdot w = u^2r + ru^2 + 2uru$. Since R is 2-torsion free we arrive at $u^2 = uRu = 0$ for all $u \in U$. A second application of [4, Lemma 1] yields $U = 0$ which is a contradiction. Thus $P(R) = 0$ implies $P(R^+) = 0$ so that, in general, $P(R^+) \subseteq P(R)$. This completes the proof.

REMARK. If R is an associative ring and $L(R)$ its attached Lie ring, then a similar application of Theorem 1 as that used in the beginning of the proof of Theorem 3 shows that $P(R) \subseteq P(L(R))$. In general however, $P(R) \neq P(L(R))$. For if R is a commutative ring then $L(R)$ is an abelian Lie ring. Therefore $L(R)$ contains no prime ideals so that $P(L(R)) = R$. On the other hand $P(R)$ consists of the set of nilpotent elements of R. Thus, if R is not a nil ring then $P(R) \subsetneq P(L(R))$. However, R always contains a subring A such that $P(R) = P(L(A))$. In fact one can always choose $A = P(R)$. For clearly $P(L(P(R))) \subseteq P(R)$. On the other hand, if $r \in P(R)$ then r is strongly nilpotent in R, hence strongly nilpotent in $L(R)$. Moreover, since $r \in P(R)$ then r is strongly nilpotent in $L(P(R))$. Therefore, $r \in P(L(P(R)))$ and it follows that $P(R) = P(L(P(R)))$.

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