PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 56, April 1976

SMOOTHNESS PROPERTIES OF GENERALIZED CONVEX FUNCTIONS

R. A. ZALIK

ABSTRACT. We present a concise and elementary proof of a theorem of Karlin and Studden concerning the smoothness properties of functions belonging to a generalized convexity cone.

In [1, Chapter XI], Karlin and Studden showed that a function which is convex with respect to an extended complete Tchebycheff system has a continuous derivative of order $n-1$, a fact which is of considerable importance in the theory of generalized convexity. Since their original proof is rather technical, we present here an alternative one which has the advantage of being very concise and elementary.

Given a function y, $y^{(k)}$ will denote its convolution with the Gauss kernel, i.e.

$$y^{(k)}(t) = \int_a^b y(s)G_k(t-s)\,ds, \quad \text{where } G_k(s) = (k/\sqrt{2\pi})\exp(-\tfrac{1}{2}k^2 a^2).$$

The set of functions convex with respect to the system $\{y_0, \ldots, y_n\}$ will be denoted by $C(y_0, \ldots, y_n)$. The abbreviations T, CT, WT and ECT will respectively stand for Tchebycheff, complete Tchebycheff, weak Tchebycheff and extended complete Tchebycheff. For the definition of other terms and symbols employed, the reader is referred to the monograph by Karlin and Studden [1, Chapters I and XI].

LEMMA 1. *Let $\{y_i\}_{i=0}^n$ be a system of continuous functions of bounded variation on an interval $[a,b]$ such that $y_0 \equiv 1$, and $\{y_i\}_{i=0}^r$ is a WT-system thereon for $r = 1, \ldots, n$. If P is the set of points of (a,b) at which all the functions y_i are differentiable, the system $\{y_i'\}_{i=1}^n$ is a WT-system on P.*

PROOF. If the functions $\{y_i\}_{i=0}^n$ are linearly dependent, the assertion is obvious. Otherwise, from the basic composition formula (cf. [1, pp. 14, 15]), we know that $\{y_i^{(k)}\}_{i=0}^n$ is an ECT-system for any natural number $k > 0$. From [1, Chapter XI, Theorem 1.2 and Remark 1.2], we conclude that also the reduced system $\{[y_i^{(k)}/y_0^{(k)}]'\}_{i=1}^n$ is an ECT-system. However, bearing in mind that $y_0 \equiv 1$, from [2, Chapter X, Exercise 9], we conclude that $\lim_{k\to\infty}[y_i^{(k)}(t)/y_0^{(k)}(t)]' = y_i'(t)$ on P, for $i = 1, \ldots, n$, whence the conclusion follows. Q.E.D.

REMARK. Lemma 1 furnishes a much shorter proof of [3, Theorem 3] in a more general framework.

Received by the editors February 20, 1975.
AMS (MOS) subject classifications (1970). Primary 26A51; Secondary 41A50.

LEMMA 2. *Let* $\{y_0, y_1, \ldots, y_{n+1}\}$ *be a WT-system on the interval* (a,b). *Then*:

(a) *If* $n \geqslant 1$ *and* $\{y_0, \ldots, y_n\}$ *is a CT-system on* (a,b), *then* y_{n+1} *is continuous on* (a,b).

(b) *If* $n \geqslant 2$ *and* $\{y_0, \ldots, y_n\}$ *is an ECT-system on* $[a,b]$, *then* y_{n+1} *has a continuous derivative on* (a,b).

PROOF. The hypotheses imply that without any loss of generality, we can assume $y_0 \equiv 1$. It will suffice to carry out the proof for every interval of the form (a,a'), $a < a' < b$. Let $D(y_0, \ldots, y_r/t_0, \ldots, t_r)$ denote the determinant of the matrix $(y_i(t_j); i,j = 0, \ldots, r)$. Let $a' < s_0 < \cdots < s_n < b$, and define

$$u_i(t) = D(y_0, \ldots, y_n/s_0, \ldots, s_{i-1}, t, s_{i+1}, \ldots, s_n),$$

$i = 0, \ldots, n$, and

$$u_{n+1}(t) = D(y_0, \ldots, y_{n+1}/s_0, \ldots, s_n, t).$$

Since

$$D(u_0, \ldots, u_n/s_0, \ldots, s_n) = \prod_{i=0}^{n} u_i(s_i) > 0,$$

it is easily seen that $\{u_0, \ldots, u_n\}$ is a T-system on (a,b), and that $u_{n+1} \in C(u_0, \ldots, u_n) = C(y_0, \ldots, y_n)$ (cf. [4, Lemma 2]). It will suffice to prove the assertions for the function u_{n+1}. Let $a < t_0 < t_1 < t_2 < a'$. Then

$$\begin{aligned} 0 &\leqslant D(u_0, \ldots, u_{n+1}/t_0, t_1, t_2, s_2, \ldots, s_n) \\ &= D(u_0, u_1, (-1)^{n-1} u_{n+1}, u_2, \ldots, u_n/t_0, t_1, t_2, s_2, \ldots, s_n) \\ &= \left(\prod_{i=2}^{n} u_i(s_i)\right) D(u_0, u_1, (-1)^{n-1} u_{n+1}/t_0, t_1, t_2). \end{aligned}$$

We therefore conclude that $\{u_0, u_1, (-1)^{n-1} u_{n+1}\}$ is a WT-system on (a,a'). In similar fashion we see that $\{u_0, u_1\}$ is a CT-system thereon, i.e. u_1/u_0 is strictly increasing. Thus the function $h = (-1)^{n-1}(u_{n+1}/u_0) \circ (u_1/u_0)^{-1}$ is convex, and thus continuous, whence the proof of (a) follows. In order to prove (b), note that, since h is convex, the function $u_{n+1} = (-1)^{n-1}[h \circ (u_2/u_1)]u_0$ admits of a representation of the form

$$u_{n+1}(t) = u_{n+1}(c) + u_0(t) \int_c^t q(s)\,ds, \qquad a < c < a', \tag{1}$$

on (a,a'), where q is left continuous on (a,a'), and continuous on a dense subset P thereof. Since $u_{n+1} \in C(y_0, \ldots, y_n)$ and is of bounded variation on every closed subinterval of (a,a'), from Lemma 1 we conclude that $\{y_1', y_2', \ldots, y_n', u_{n+1}'\}$ is a WT-system on P. Since $u_{n+1}' = q$ on P, and q is left continuous, we conclude that $\{y_1', y_2', \ldots, y_n', q\}$ is a WT-system on (a,a'). The hypotheses imply that $\{y_1', y_2', \ldots, y_n'\}$ is a CT-system on (a,b). Thus part (a) of this lemma implies that q is continuous on (a,a'), and the conclusion follows from (1). Q. E. D.

THEOREM. *Assume that* $\{y_i\}_{i=0}^{n}$ *is an extended complete Tchebycheff system on* $[a,b]$, *and let* $y \in C(y_0, \ldots, y_n)$. *Then*:

(a) *The function y belongs to the continuity class $C^{n-1}(a,b)$.*

(b) *The function $(D_{n-2}\cdots D_0 y)(t)/w_{n-1}(t)$ has a right continuous derivative on (a,b), and $(D_{n-1}^{(R)} D_{n-2}\cdots D_0 y)(t)/w_n(t)$ is increasing thereon* (*cf.* [1, *Chapter* XI, *Theorem* 11.1 *and Theorem* 2.1]).

PROOF. (a) The cases $n = 1$ and $n = 2$ have been proved in the preceding lemma. Assume the assertion to be true for $n = k$, and let $n = k + 1$. From the preceding lemma we know that $y \in C^1(a,b)$. Dividing the functions y_i and y by y_0 and differentiating, by Lemma 1 we conclude that $D_0 y \in C(D_0 y_1, \ldots, D_0 y_n)$. Since by inductive hypothesis, $D_0 y \in C^{n-2}(a,b)$, the conclusion follows.

(b) By repeated application of Lemma 1, we see that

$$\left\{w_{n-1}(t), w_{n-1}(t)\int_a^t w_n(s)\,ds, (D_{n-2}\cdots D_0 y)(t)\right\}$$

is a weak Tchebycheff system. Dividing by w_{n-1} and making the change of variable $r = \int_a^t w_n(s)\,ds$, the conclusion follows from the properties of convex functions. Q. E. D.

BIBLIOGRAPHY

1. S. Karlin and W. J. Studden, *Tchebycheff systems: With applications in analysis and statistics*, Pure and Appl. Math., vol. 15, Interscience, New York, 1966. MR **34** #4757.

2. I. P. Nathanson, *Theory of functions of a real variable*, vol. II, GITTL, Moscow, 1957; English transl. of 2nd rev. ed., Ungar, New York, 1961. MR **26** #6309.

3. G. Mühlbach, *A recurrence formula for generalized divided differences and some applications*, J. Approximation Theory **9** (1973), 165–172. MR **50** #6106.

4. R. A. Zalik, *Existence of Tchebycheff extensions*, J. Math. Anal. Appl. **51** (1975), 68–75.

DEPARTMENT OF MATHEMATICS, BEN GURION UNIVERSITY OF THE NEGEV, P.O.B. 2053, BEERSHEVA, ISRAEL