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# THE FINAL VALUE PROBLEM FOR SOBOLEV EQUATIONS

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ABSTRACT. Let $A$ and $B$ be $m$-accretive linear operators in a complex Hilbert space $H$ with $D(A) \subset D(B)$. The method of quasi-reversibility is used to obtain a solution to the Sobolev equation $(d/dt)[(I + B)u(t)] + Au(t) = 0$, $0 < t < 1$, which approximates a specified final value $u(1) = f$. In general, when $D(A) \subset D(B)$, it is not possible to find a solution which achieves exactly the final value $u(1) = f$.

1. Let $A$ and $B$ be a linear $m$-accretive operators in a complex Hilbert space $H$ with $D(A) \subset D(B)$. The purpose of the present note is to show how the method of quasi-reversibility [4] can be used to treat the final value problem

$$Lu = (d/dt)[(I + B)u(t)] + Au(t) = 0, \qquad 0 < t < 1, \tag{1.1}$$

$$u(1) = f. \tag{1.2}$$

Since this problem is not well posed, in general, when $D(A) \subset D(B)$, one may consider instead the problem of approximation of the final value, that is, given $\rho > 0$, find, if possible, a solution $u_\rho$ of (1.1) such that $\|u_\rho(1) - f\| < \rho$. Quasi-reversibility is a constructive method of determining such a solution.

In this method, one approximates the operator $L$ by a nearby operator $L_\rho$ such that the final value problem for $L_\rho$ is well posed (although the initial value problem may be ill posed; hence the term quasi-reversibility). The value $v(0)$ of the solution of $L_\rho v = 0$, $v(1) = f$, is then used as an initial value in solving (1.1).

Of course, various approximating operators $L_\rho$ may be used. Here we approximate (1.1) by

$$L_\rho v = (d/dt)[(I + B + \varepsilon A)v(t)] + Av(t) = 0, \quad \varepsilon = \varepsilon(\rho). \tag{1.3}$$

For this choice of $L_\rho$ both the initial and final value problems are well posed. Furthermore, this type of approximation is stable in a sense to be made precise.

Our choice of (1.3) is suggested by the results of [6] where such an approximation procedure is used to treat the special case $B = 0$. In fact, we shall show how the results of [6] can be used to obtain estimates in the general case as well.

An additional condition imposed on the operators $A$ and $B$ is a sector condition:

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$$(1.4)\qquad |\arg(Ax,(I+B)x)| \leqslant \pi/4, \quad \forall x \in D(A).$$

In §3 we shall give examples of how operators may be constructed which satisfy (1.5). When $B = 0$, (1.4) is equivalent to a hypothesis that the semigroup generated by $-A$ has an analytic extension into the sector $|\arg z| < \pi/4$ of the complex plane.

2. We first consider the case $B = 0$. $A$ is assumed to be $m$-accretive, that is, $\operatorname{Re}(Ax,x) \geqslant 0$ for all $x \in D(A)$ and $\operatorname{Rg}(I + A) = H$. By a *solution* of

$$(2.1)\qquad Lu = du/dt + Au = 0$$

on $[0, 1]$ is meant a function $u \in C([0,1]; H) \cap C'((0,1); H)$ such that for all $t$ in $(0, 1)$, $u(t) \in D(A)$ and (2.1) is satisfied.

Let $S(t)$, $t \geqslant 0$, be the continuous semigroup of contractions on $H$ generated by $-A$ and, for each $\varepsilon > 0$, let $S_\varepsilon(t)$, $-\infty < t < +\infty$, be the continuous group of bounded operators on $H$ generated by the bounded, dissipative operator $-A_\varepsilon = \varepsilon^{-1}((I + \varepsilon A)^{-1} - I)$. Let $f \in H$ and set $v(t) = S_\varepsilon(t - 1)f$. Then $v$ satisfies $dv/dt + A(I + \varepsilon A)^{-1}v = 0$ and so is "formally" (that is, if $v \in D(A)$ and the interchange of operations is justified) a solution of the problem

$$(2.2)\qquad (d/dt)[(1 + \varepsilon A)v(t)] + Av(t) = 0,\ t < 1, \quad v(1) = f.$$

Let $u_\varepsilon(t)$ be the solution on $[0, 1]$ of (2.1) satisfying the initial condition $u_\varepsilon(0) = S_\varepsilon(-1)f$. Then $u_\varepsilon(t) = S(t)S_\varepsilon(-1)f$ and one expects $u_\varepsilon(1)$ to approximate $f$ in some sense. The following results are proved in [6]: Let $E_\varepsilon(t) = S(t)S_\varepsilon(-t)$, $t \geqslant 0$, and assume $A$ is *m-sectorial* with semiangle $\pi/4$ (that is, (1.4) holds with $B = 0$). Then

(I) $E_\varepsilon(t)$, $t \geqslant 0$, is a contraction semigroup on $H$ and $E_\varepsilon(t)f \to f$ as $\varepsilon \to 0_+$ for each $f \in H$, uniformly on bounded intervals of $t$. Furthermore

$$\|E_\varepsilon(t)f - f\| \leqslant t\|Af - A_\varepsilon f\|, \qquad f \in D(A),$$
$$\|E_\varepsilon(t)f - f\| \leqslant \varepsilon t\|A^2 f\|, \qquad f \in D(A^2).$$

(II) For each $f \in H$, (2.1) has at most one solution on $[0, 1]$ satisfying $u(1) = f$. Suppose $f = S(1)\xi$ for some (necessarily unique) $\xi \in H$. Then the final value problem has a solution $u(t) = S(t)\xi$ on $[0, 1]$ and for $m = 0, 1, \ldots,$

$$\|u_\varepsilon^{(m)}(t) - u^{(m)}(t)\| \leqslant (M/t)^m \|E_\varepsilon(1)\xi - \xi\|, \qquad \varepsilon > 0,\ 0 < t \leqslant 1,$$
$$\|u_\varepsilon^{(m)}(t) - u^{(m)}(t)\| \leqslant \varepsilon[M/(t - \delta)]^m \|A^2 S(\delta)\xi\|,$$
$$\varepsilon > 0,\ 0 < \delta < 1,\ \delta < t \leqslant 1,$$

where $M$ is a positive constant.

Now we turn to the general case $B \neq 0$. $A$ and $B$ are assumed $m$-accretive with $D(A) \subset D(B)$. By a *solution* of (1.1) on $[0, 1]$ is meant a function $u\colon [0, 1] \to D(B)$ such that $(I + B)u \in C([0, 1]; H) \cap C'((0, 1); H)$ and for all $t$ in $(0, T)$, $u(t) \in D(A)$ and (1.1) is satisfied. Note that the definition requires that $u(1) \in D(B)$.

Let $\tilde{B}$ denote the restriction of $B$ to $D(A)$ and set

$$\tilde{A} = A(I + \tilde{B})^{-1}, \quad D(\tilde{A}) = \mathrm{Rg}\,(I + \tilde{B}).$$

One verifies that a function $u$ is a solution of (1.1) on $[0, 1]$ if and only if $\tilde{u} = (I + B)u$ is a solution on $[0, 1]$ of

$$d\tilde{u}/dt + \tilde{A}\tilde{u} = 0. \tag{2.3}$$

If $\mathrm{Re}(Ax, (I + B)x) \geqslant 0$, $\forall x \in D(A)$, then $\tilde{A}$ is accretive and, moreover, $m$-accretive ([5]; c f. [3]). If the more restrictive condition (1.4) is satisfied, then $\tilde{A}$ is $m$-sectorial with semiangle $\pi/4$.

Assume that (1.4) holds and let $\tilde{S}(t)$, $t \geqslant 0$, be the analytic semigroup of contractions on $H$ generated by $-\tilde{A}$, and $\tilde{S}_\varepsilon(t)$, $-\infty < t < t + \infty$, be the group of bounded operators on $H$ generated by $-\tilde{A}_\varepsilon = \varepsilon^{-1}((I + \varepsilon\tilde{A})^{-1} - I)$. If $f \in D(B)$, the function $\tilde{v}(t) = \tilde{S}_\varepsilon(t - 1)(I + B)f$ is formally a solution on $[0, 1]$ of (2.2) with $A$ replaced by $\tilde{A}$, and $\tilde{v}$ satisfies $\tilde{v}(1) = (I + B)f$. Hence

$$v(t) = (I + B)^{-1}\tilde{S}_\varepsilon(t - 1)(I + B)f$$

is formally a solution on $[0, 1]$ of

$$(d/dt)[(I + B + \varepsilon A)v(t)] + Av(t) = 0$$

such that $v(1) = f$. Thus we define $u_\varepsilon(t)$ to be the solution of (1.1) on $[0, 1]$ satisfying the initial condition $u_\varepsilon(0) = (I + B)^{-1}\tilde{S}_\varepsilon(-1)(I + B)f$, that is

$$u_\varepsilon(t) = (I + B)^{-1}\tilde{S}(t)\tilde{S}_\varepsilon(-1)(I + B)f.$$

THEOREM 2.1. *Let $A$ and $B$ be $m$-accretive operators with $D(A) \subset D(B)$ satisfying* (1.4) *and suppose $f \in D(B)$. Then $u_\varepsilon(1) \to f$ as $\varepsilon \to 0_+$ and the approximation procedure is stable in the sense that*

$$\|(I + B)u_\varepsilon(1)\| \leqslant \|(I + B)f\| \quad \textit{for all } \varepsilon > 0.$$

*Furthermore,*

$$\|u_\varepsilon(1) - f\| \leqslant \varepsilon\|\tilde{A}_\varepsilon Af\|, \qquad f \in D(A),$$
$$\|u_\varepsilon(1) - f\| \leqslant \varepsilon\|\tilde{A}Af\|, \qquad f \in D(\tilde{A}A).$$

PROOF. These results follow from (I) above as applied to (2.3). For example, if $f \in D(A)$ we have, since $B$ is accretive,

$$\begin{aligned}\|u_\varepsilon(1) - f\| &\leqslant \|(I + B)(u_\varepsilon(1) - f)\| = \|\tilde{S}(1)\tilde{S}_\varepsilon(-1)(I + B)f - (I + B)f\| \\ &\leqslant \|\tilde{A}(I + B)f - \tilde{A}_\varepsilon(I + B)f\| = \|Af - (I + \varepsilon\tilde{A})^{-1}Af\| \\ &= \varepsilon\|\tilde{A}_\varepsilon Af\|.\end{aligned}$$

Similarly, we deduce the following from (II):

THEOREM 2.2. *With the hypotheses of Theorem* 2.1, (1.1) *has at most one* *solution on* $[0,1]$ *satisfying* $u(1) = f$. *Suppose* $f = (I+B)^{-1}\tilde{S}(1)(I+B)\xi$ *for some (necessarily unique)* $\xi \in D(B)$. *Then the final value problem has a solution* $u(t)$ *on* $[0,1]$ *and for* $m = 0, 1, \ldots,$

$$\|u_\varepsilon^{(m)}(t) - u^{(m)}(t)\| \leqslant (M/t)^m \|\tilde{E}_\varepsilon(1)(I+B)\xi - (I+B)\xi\|, \qquad \varepsilon > 0,\ 0 < t \leqslant 1,$$

$$\|u_\varepsilon^{(m)}(t) - u^{(m)}(t)\| \leqslant [M/(t-\delta)]^m \|\tilde{A}^2\tilde{S}(\delta)(I+B)\xi\|, \qquad \varepsilon > 0,\ 0 < \delta < 1,\ \delta < t \leqslant 1,$$

*where* $\tilde{E}_\varepsilon(t) = \tilde{S}(t)\tilde{S}_\varepsilon(-t)$ *and* $M$ *is a positive constant.*

PROOF. The function $\tilde{u}(t) = \tilde{S}(t)(I+B)\xi$ is a solution of (2.3) on $[0,1]$ satisfying $\tilde{u}(1) = (I+B)f$; hence $u(t) = (I+B)^{-1}\tilde{u}(t)$ is a solution of (1.1), (1.2) on $[0,1]$. Since $\tilde{S}(t)$ is an analytic semigroup, $\tilde{u} \in C^\infty((0,1];H)$, hence $u \in C^\infty((0,1];D(B))$ where $D(B)$ is normed with its graph norm. It follows easily that the strong $H$-derivatives, $u^{(m)}(t)$, all belong to $D(B)$ and $(I+B)u^{(m)}(t) = ((I+B)u(t))^{(m)}$. Hence,

$$\|u_\varepsilon^{(m)}(t) - u^{(m)}(t)\| \leqslant \|(I+B)(u_\varepsilon^{(m)}(t) - u^{(m)}(t))\| = \|\tilde{u}_\varepsilon^{(m)}(t) - \tilde{u}^{(m)}(t)\|.$$

The estimates therefore follow from (II) above.

3. In this section we shall show how $m$-accretive operators $A$ and $B$ satisfying the sector condition (1.4) may be constructed.

Let $C$ be a selfadjoint operator and $E(\lambda)$, $-\infty < \lambda < +\infty$, be the corresponding resolution of the identity. A spectral measure $E$ is then determined by setting $E((\lambda_1,\lambda_2]) = E(\lambda_2) - E(\lambda_1)$. Let $f(\lambda)$ and $g(\lambda)$ be complex valued Baire functions defined and finite $E$-almost everywhere on the real line (that is, except at most on a set of measure zero with respect to the spectral measure $E$). One may then define operators $A$ and $B$ by setting

$$A = \int_{-\infty}^{\infty} f(\lambda)E(d\lambda), \quad B = \int_{-\infty}^{\infty} g(\lambda)E(d\lambda)$$

with

$$D(A) = \left\{x: \int_{-\infty}^{\infty} |f(\lambda)|^2(E(d\lambda)x,x) < \infty\right\},$$

$$D(B) = \left\{x: \int_{-\infty}^{\infty} |g(\lambda)|^2(E(d\lambda)x,x) < \infty\right\}.$$

THEOREM 3.1. *Assume the following hold for all* $\lambda$ *in the spectrum of* $C$:

(i) $\operatorname{Re} f(\lambda) \geqslant 0$, $\operatorname{Re} g(\lambda) \geqslant 0$,

*Then* $A$ *and* $B$ *are m-accretive operators satisfying the sector condition* (1.4) *for* $x \in D(A) \cap D(B)$.

PROOF. We need only apply the operational calculus of selfadjoint operators [1, Chapter XII]. Since $(E(d\lambda)x,x)$ determines a positive measure,

$$\operatorname{Re}(Ax,x) = \int_{-\infty}^{\infty} \operatorname{Re} f(\lambda)(E(d\lambda)x,x), \quad x \in D(A),$$

is accretive if $\operatorname{Re} f(\lambda) \geqslant 0$. In addition, $A$ is closed with dense domain and

$$\operatorname{Re}(A^* x, x) = \int_{-\infty}^{\infty} \operatorname{Re} \bar{f}(\lambda)(E(d\lambda)x, x), \qquad x \in D(A^*).$$

Thus both $A$ and its adjoint are accretive operators, and this is sufficient to conclude that $A$ is $m$-accretive. Similary for $B$.

We also have, for $x \in D(A) \cap D(B)$,

$$(Ax, (I + B)x) = \int_{-\infty}^{\infty} f(\lambda)(1 + \bar{g}(\lambda))(E(d\lambda)x, x).$$

Thus $|\arg(Ax, (I + B)x)| \leqslant \pi/4$ if

$$|\arg(f(\lambda)(1 + \bar{g}(\lambda)))| = |\arg f(\lambda) - \arg(1 + g(\lambda))| \leqslant \pi/4.$$

Of course, one also has $D(A) \subset D(B)$ if, for example, $|f(\lambda)| \geqslant |g(\lambda)|$ on the spectrum of $C$.

The operators $A$ and $B$ just constructed are known to be normal operators. Other types of $m$-accretive operators which satisfy (1.4) may be constructed from fractional powers of an $m$-accretive operator $C$ as follows: Let $0 < \alpha < 1$ and $C^\alpha$ denote the indicated fractional power of $C$; if $x \in D(C)$ then

$$C^\alpha x = \frac{\sin \pi\alpha}{\pi} \int_0^{\infty} \lambda^{1-\alpha}(C + \lambda)^{-1} Cx \, d\lambda.$$

The following properties of $C$ are well known (see [2], [7]): (1) $C^\alpha$ is $m$-sectorial with semiangle $\pi\alpha/2$. (2) $D(C^\beta) \subset D(C^\alpha)$ if $\alpha < \beta$. (3) $C^{\alpha+\beta}x = C^\alpha(C^\beta x)$ if $x \in D(C^2)$, $\alpha + \beta < 1$. (4) $C^\alpha$ commutes with every bounded operator that commutes with $C$.

Let $M$ and $N$ be positive integers and $\{\alpha_n\colon 1 \leqslant n \leqslant N\}$ and $\{\beta_n\colon 1 \leqslant n \leqslant M\}$ be real numbers such that $\alpha_N \geqslant \beta_M$ and

$$0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_N \leqslant 1, \tag{3.1}$$

$$0 \leqslant \beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_M \leqslant 1. \tag{3.2}$$

Set

$$A = \sum_{n=1}^{N} a_n C^{\alpha_n}, \quad D(A) = D(C^{\alpha_N}), \quad \forall a_n \geqslant 0, \tag{3.3}$$

$$B = \sum_{n=1}^{M} b_n C^{\beta_n}, \quad D(B) = D(C^{\beta_M}), \quad \forall b_n \geqslant 0. \tag{3.4}$$

$A$ and $B$ are sectorial operators with respective semiangles $\pi\alpha_N/2$ and $\pi\beta_M/2$, $D(A) \subset D(B)$ and for $x \in D(A)$,

$$(Ax, (I + B)x) = \sum_{n,m} a_n(1 + b_m)(C^{\alpha_n}x, C^{\beta_m}x).$$

If $x \in D(C^2)$, then $(C^{\alpha_n}x, C^{\beta_m}x)$ belongs to a sector $|\arg z| \leqslant (\pi/2)|\alpha_n - \beta_m|$ as can be seen by writing, for example,

$$(C^{\alpha_n}x, C^{\beta_m}x) = (C^{\alpha_n - \beta_m}C^{\beta_m}x, C^{\beta_m}x), \qquad \alpha_n > \beta_m.$$

Thus if $x \in D(C^2)$,

$$|\arg(Ax, (I + B)x)| \leqslant \pi\theta/2 \tag{3.5}$$

where $\theta = \max_{n,m}|\alpha_n - \beta_m|$.

Suppose $x \in D(A)$ and set $x_k = k^2(C + k)^{-2}x$. Then $x_k \in D(C^2)$, $x_k \to x$ and for $0 < \alpha \leqslant \alpha_N$, $C^\alpha x_k = k^2(C + k)^{-2}C^\alpha x \to C^\alpha x$ as $k \to \infty$. Thus $Ax_k \to Ax$, $Bx_k \to Bx$ and therefore (3.5) holds for each $x \in D(A)$. One also sees in the same way that $(a_n C^{\alpha_n}x, a_m C^{\alpha_m}x)$ lies in the right-half of the complex plane. Since $a_n C^{\alpha_n}$ and $a_m C^{\alpha_m}$ are $m$-accretive, it follows from [5] that the same is true for their sum. A simple induction argument then shows that $A$ and $B$ are $m$-accretive. We have proved

THEOREM 3.2. *Suppose* $\{\alpha_n\}$, $\{\beta_n\}$ *satisfy* (3.1), (3.2),

$$\alpha_N \geqslant \beta_N \quad \textit{and} \quad \max_{n,m}|\alpha_n - \beta_m| \leqslant \tfrac{1}{2}.$$

*Then A and B, defined by* (3.3), (3.4) *are m-accretive operators which satisfy* (1.4).

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