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LIE GROUPS ISOMORPHIC TO DIRECT PRODUCTS OF UNITARY GROUPS

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ABSTRACT. A criterion is given for a compact connected subgroup of $\mathrm{Gl}(n, \mathbf{C})$ to be isomorphic to a direct product of unitary groups. It implies that a compact connected subgroup of rank n in $\mathrm{Gl}(n, \mathbf{C})$ is isomorphic to a direct product of unitary groups.

The paper gives a generalization of some of the results in [3]. Let G be a compact connected subgroup of $\mathrm{Gl}(n, \mathbf{C})$. We denote by $L(G)$ the Lie algebra of G and set $H(G) = iL(G)$. The rank of G is the dimension of a maximal torus in G (see [1, p. 93]).

THEOREM. *Let G be a compact connected subgroup of rank k in* $\mathrm{Gl}(n, \mathbf{C})$. *Suppose there exist $r \geqslant k$ orthogonal idempotents $a_1, \ldots, a_r$ in $H(G)$. Then $r = k$ and G is isomorphic (as a Lie group) to a direct product of unitary groups: $G \cong U(n_1) \times \cdots \times U(n_m)$ with $n_1 + \cdots + n_m = k$.*

PROOF. By [2, p. 176, Theorem 1] G is similar to a subgroup of $U(n)$. Hence we may assume that G is a subgroup of $U(n)$. Thus the operators in $H(G) \subset \mathrm{End}(\mathbf{C}^n)$ are hermitian. Since $a_1, \ldots, a_r$ commute we see that $T = \{\exp(it_1 a_1 + \cdots + it_r a_r) | t_1, \ldots, t_r \in \mathbf{R}\}$ is a torus in G of dimension r. Clearly $r = k$ and T is a maximal torus. If $a \in H(G)$ then $\exp(ita) \in G$ $(t \in \mathbf{R})$ and is contained in some conjugate of T (see [1, p. 89]), i.e. $\exp(ita) \in u^{-1}Tu = u^*Tu$ for some $u \in G$. It follows that $a = t_1 u^* a_1 u + \cdots + t_r u^* a_r u$. Since $a^2 = t_1^2 u^* a_1 u + \cdots + t_r^2 u^* a_r u$ and $u^* a_s u \in H(G)$ for $s = 1, \ldots, r$ we see that $a^2 \in H(G)$. Let $b \in H(G)$, too. Since $ab + ba = (a + b)^2 - a^2 - b^2$ we see that $ab + ba \in H(G)$. Also, $ab - ba \in iH(G)$ since $ia, ib \in L(G)$. Thus $ab \in H(G) + iH(G)$. Let $A(G) = H(G) + iH(G)$. It follows that $A(G)$ is an algebra. Clearly, it is a finite dimensional C^*-algebra. By the Wedderburn decomposition there exist central idempotents $e_1, \ldots, e_m \in A(G) = A$ such that $A = Ae_1 \oplus \cdots \oplus Ae_m$ and Ae_s is isomorphic to $\mathrm{End}(X_s)$ for some finite dimensional vector space X_s over $\mathbf{C}$ $(s = 1, \ldots, m)$.

The ideal Ae_s is closed, hence selfadjoint and a C^*-subalgebra of A. Clearly, e_s is the identity on Ae_s and hence $e_s^* = e_s$. Consider the group V of unitary elements in Ae_s. The isomorphism $Ae_s \cong \mathrm{End}(X_s)$ defines a (continuous) representation of V on X_s. Using once more [2, p. 176, Theorem 1] we equip X_s with an inner product such that the isomorphism maps V into the unitary

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group of $\mathfrak{L}(X_s)$, the C^*-algebra of all linear operators on the Hilbert space X_s. Consequently, hermitian elements in Ae_s are mapped into hermitian operators and our isomorphism in an isometric*-isomorphism. We identify the algebras Ae_s and $\mathfrak{L}(X_s)$ in this sense.

Since exp: $L(G) \to G$ is surjective, $G \subset A$. If $u \in G$ then $(ue_s)^* ue_s = e_s u^* ue_s = e_s$. Thus ue_s is a unitary operator on X_s. Consider the smooth homomorphism $G \to U(X_1) \times \cdots \times U(X_m)$ given by $u \mapsto (ue_1, \ldots, ue_m)$ ($U(X_s)$ denotes the unitary group on X_s). We claim this homomorphism is onto. Let $u_1 \in U(X_1)$. There exists a hermitian element $h_1 \in Ae_1$ such that $\exp(ih_1) = u_1$. Consider h_1 as an element in A. Then $\exp(ih_1) = (u_1, 1, \ldots, 1)$. Observe that the inverse $(ue_1, \ldots, ue_m) \mapsto ue_1 + \cdots + ue_m$ is also smooth and that $\operatorname{rank}(U(n_1) \times \cdots \times U(n_m)) = n_1 + \cdots + n_m$.

COROLLARY. *Let G be a compact connected subgroup of rank n in* $\mathrm{Gl}(n, \mathbf{C})$. *Then G is isomorphic (as a Lie group) to a direct product of unitary groups.*

PROOF. As before, we may assume that $G \leqslant U(n)$. Let T be a maximal torus in G. Then $iL(T)$ contains n commuting linearly independent hermitian operators, say $h_1, \ldots, h_n$. It is well known that these operators have a common orthogonal eigenbasis. Thus there exist $s \leqslant n$ orthogonal projections $p_1, \ldots, p_s$ such that every h_i is a linear combination of $p_1, \ldots, p_s$. Since $h_1, \ldots, h_n$ are linearly independent, $s = n$. Thus $iL(G)$ contains n orthogonal idempotents and we may use the Theorem.

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