ISOPERIMETRIC INEQUALITIES FOR A NONLINEAR EIGENVALUE PROBLEM

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Abstract. An estimate for the spectrum of the two-dimensional eigenvalue problem \( \Delta u + \lambda e^u = 0 \) in \( D (\lambda > 0) \), \( u = 0 \) on \( \partial D \) is derived, and upper and lower pointwise bounds for the solutions are constructed.

1. Let \( D \) be a simply connected bounded domain in the plane with a piecewise analytic boundary \( \partial D \). Consider the nonlinear Dirichlet problem

\[
\begin{align*}
\Delta u(x) + \lambda e^{u(x)} &= 0 & \text{in } D, \\
u(x) &= 0 & \text{on } \partial D,
\end{align*}
\]

where \( \lambda \) is a positive real number and \( x \) stands for the generic point \( (x_1, x_2) \). This problem arises in the theory of self-ignition of a chemically active gas [4 and the literature cited there] and has been studied by many authors [2], [3], [5].

It was shown in [3] and [5] that there exists a number \( \lambda^* > 0 \) such that the problem has at least one solution for each \( \lambda \leq \lambda^* \), but does not have solutions for \( \lambda > \lambda^* \). Bounds for \( \lambda^* \) are found in [2]. In particular it was proved that \( \lambda^* > 2\pi/A \) where \( A \) denotes the area of \( D \). Equality is attained if and only if \( D \) is a circle. In this paper we prove that \( \lambda^* < 2/R_0^2 \), \( R_0 \) being the maximal conformal radius of \( D \). We also give estimates for the solutions by means of the conformal radius. Our proofs are based on the introduction of a special system of coordinates defined by the level lines; see [6].

2. Let \( g(x, \xi) \) be the Green’s function for the Laplace operator, vanishing on \( \partial D \). It is well known that for fixed \( x \in D \)

\[
g(x, \xi) = (2\pi)^{-1} \log(R_x/|x - \xi|) + H(x, \xi)
\]

where \( R_x \) is the conformal radius of \( x \) with respect to \( D \),

\[
|x - \xi| = \left( \sum_{i=1}^{2} (x_i - \xi_i)^2 \right)^{1/2}
\]

and \( H(x, \xi) \) is a harmonic function of \( \xi \) with \( \lim_{\xi \to x} H(x, \xi) = 0 \). With the help of this Green’s function Problem (1) can be written as an integral equation:

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We now keep $x$ fixed and denote by $D(t)$ the domain $\{ \xi \in D; g(x, \xi) > t \}$. It is homeomorphic to a circle. Let us assume that Problem (1) has a solution $u \in C^2(D) \cap C^0(D)$. Since $e'$ is real analytic, $u(x)$ is also real analytic. Define

$$a(t) = \int_{D(t)} e^{u(\xi)} d\xi.$$

Let $\Gamma(t) = \{ \xi \in D; g(x, \xi) = t \}$. It is a simple closed curve and analytic for all $t > 0$. Denote by $\delta n > 0$ the piece of normal between $\Gamma(t)$ and $\Gamma(t + dt)$. If $s$ is the arclength of $\Gamma(t)$, then [6, especially p. 213]

$$da(t) = a(t + dt) - a(t) = - \oint_{\Gamma(t)} e^{u(\xi)} \delta n ds_\xi + o(dt).$$

Because of the strong maximum principle, $|\text{grad } g(x, \xi)|$ cannot vanish on $\Gamma(t)$, hence

$$\frac{da}{dt} = - \oint_{\Gamma(t)} e^{u(\xi)} |\text{grad } g(x, \xi)|^{-1} ds_\xi.$$

(3) can be written in the following form:

$$u(x) = -\lambda \int_0^\infty t \cdot \frac{da}{dt} \cdot dt.$$

Integration by parts yields

$$u(x) = \lambda \int_0^\infty a(t) dt.$$

By the Schwarz inequality we have

$$\oint_{\Gamma(t)} e^{u(\xi)} |\text{grad } g|^{-1} d\xi \cdot \oint_{\Gamma(t)} |\text{grad } g| d\xi_\xi \geq \left\{ \oint_{\Gamma(t)} e^{u(\xi)/2} ds_\xi \right\}^2$$

and therefore

$$-\frac{da}{dt} \geq \left\{ \oint_{\Gamma(t)} e^{u(\xi)/2} ds_\xi \right\}^2.$$

Consider the abstract surface given by the domain $D \subset \mathbb{R}^2$ and the Riemann metric $ds^2 = e^{u(\xi)} ds^2$. Its Gaussian curvature is $K = -\Delta u/(2e^u) = \lambda/2$.

Because of the isoperimetric inequality for manifolds of constant Gaussian curvature [1, p. 514],

$$\left\{ \oint_{\Gamma(t)} e^{u(\xi)/2} ds \right\}^2 \geq 4\pi a(t) - \frac{\lambda}{2} a^2(t).$$

This inequality together with (6) implies that

$$-\frac{da}{dt} \geq 4\pi a(t) - \frac{\lambda}{2} a^2(t)/2.$$
Thus, \( m(t) = e^{-4\pi i (1/a(t) - \lambda / 8\pi)} \) is a nondecreasing function of \( t \). From (2) we conclude that

\[
\lim_{t \to \infty} m(t) = 1/(\pi R_x^2 e^{u(x)}).
\]

Hence

\[
m(t) \leq 1/(\pi R_x^2 e^{u(x)}) \quad \text{for all } t > 0,
\]

\[
a(t) \geq \frac{1}{e^{4\pi i / (\pi R_x^2 e^{u(x)})} + \lambda / 8\pi}.
\]

If we insert this estimate into (5) and integrate, we obtain

(7) \[ e^{u(x)/2} \geq 1 + \lambda R_x^2 e^{u(x)}/8. \]

Let us put for short \( \beta = \lambda R_x^2/8 \). Then (7) yields

(8) \[ \left[ e^{u(x)/2} - 1/(2\beta) \right]^2 + 1/\beta - 1/(4\beta^2) \leq 0. \]

Hence, the expression \( 1/\beta - 1/(4\beta^2) \) must be nonpositive, and we therefore have

(9) \[ \lambda R_x^2 \leq 2. \]

From this inequality we conclude that

(10) \[ \lambda^* \leq 2/R_0^2. \]

Consider now a circle of radius \( R \). The radially symmetric solutions of (1) are in this case [4]

\[ u_i(r) = \log \frac{b_i}{(1 + \lambda (b_i/8) r^2)^2} \]

where \( r = |x| \) and

\[ b_i = \frac{32}{\lambda^2 R^4} \left( 1 - \frac{\lambda R^2}{4} + (-1)^i \left( 1 - \frac{\lambda R^2}{2} \right)^{1/2} \right), \quad i = 1, 2. \]

the function \( u_i(r) \) corresponds to the minimal solution [5], [2]. By [5, Theorem 3.2] it follows that Problem (1) has a solution if and only if a minimal solution exists. Thus, \( \lambda^* = 2/R_0^2 \).

We therefore have proved

**Theorem 1.** Let \( D \) be a simply connected domain in \( \mathbb{R}^2 \), and let \( R_0 \) be its maximal conformal radius. Then \( \lambda^* \leq 2/R_0^2 \). Equality holds for the circle.

The next result is an immediate consequence of (8).

**Theorem 2.** Under the assumptions of Theorem 1 we have

(11) \[ 1 - \sqrt{1 - 4\beta} \leq 2e^{-u(x)/2} \leq 1 + \sqrt{1 - 4\beta} \]

where \( \beta = \lambda R_x^2 \).
Equality holds at the right-hand side if $D$ is a circle, $x$ is taken at the center and $u(x)$ is the minimal solution $u_1(r)$. Equality holds at the left-hand side if $D$ is a circle, $x$ is taken at the center and $u(x)$ corresponds to $u_2(r)$.

**Remark.** Since $R_x \neq 0$ for $x \in D, x \notin \partial D$, (11) leads to the conjecture that for fixed $\lambda$ all solutions of Problem (1) are uniformly bounded.

Let $d(x)$ be the distance from the point $x \in D$ to the boundary $\partial D$. By the monotony of $R_x$ with respect to the domain it follows that $R_x \geq d(x)$. This inequality together with (11) leads to the

**Corollary.** Under the assumptions of Theorem 1 we have

$$1 - \sqrt{1 - \lambda d^2(x)/2} \leq 2e^{-u(x)/2} \leq 1 + \sqrt{1 - \lambda d^2(x)/2}.$$ 

**References**


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