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INTEGRAL CLOSURES OF UNCOUNTABLE COMMUTATIVE REGULAR RINGS

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ABSTRACT. Necessary and sufficient conditions are given for a commutative regular ring to have a prime integrally closed extension.

In this paper we give necessary and sufficient conditions for a commutative regular ring R to have a prime integral closure. In [1] it was shown that for a commutative regular ring R to have a prime integral closure, it is necessary that every polynomial $p(x)$ in $R[x]$ have an unambiguous factor (see definitions below), and that in the case that R is countable this condition is also sufficient. An example was given to show that this condition is not sufficient if R is uncountable. It was also seen in [1] that if R has a prime integral closure, then this closure is unique. I would like to thank Bonnie Gold and Gadi Moran for helpful conversations during the preparation of this paper.

DEFINITIONS. (1) K_{CR} is the theory of commutative regular rings;

$$K_{\overline{CR}} = K_{CR} \cup \{\text{every monic polynomial has a root}\}$$

is the theory of integrally closed commutative regular rings.

(2) If $R \models K_{CR}$ and $p(x) \in R[x]$, we call $p(x)$ unambiguous if on no nonzero idempotent e is it the case that $p(x) = u(x)v(x)$ with $(u(x), v(x)) = 1$ on e. (An identity holds on e if it holds in Re.) This condition is equivalent to $p(x)$ being a power of an irreducible polynomial at every point of S_R, the Stone space of R ($=$ Spec (R)).

$$T = K_{CR} \cup \{\text{every polynomial has an unambiguous factor}\}.$$

(3) If $R \models K_{CR}$ and $R \subset \overline{R} \models K_{\overline{CR}}$, we call $\overline{R}$ a prime extension of R to a model of $K_{\overline{CR}}$, or an (in fact the) integral closure of R if whenever $f\colon R \to R_1 \models K_{\overline{CR}}$ is an embedding, f extends to an embedding of $\overline{R}$ into R_1. If we drop the condition that $\overline{R} \models K_{\overline{CR}}$ we call $\overline{R}$ a prime extension of R.

(4) If $R \models K_{CR}$ and $R \subset \overline{R} \models K_{CR}$, we call $\overline{R}$ sequentially prime over R if $\overline{R} = \bigcup_{\alpha<\lambda} R_\alpha$ with $R_0 = R$, $R_\delta = \bigcup_{\alpha<\delta} R_\alpha$ for limit ordinals $\delta < \lambda$ and $R_{\alpha+1} = R_\alpha[a_\alpha]$, with a_α a root of an unambiguous polynomial $p_\alpha(x) \in R_\alpha[x]$. (In other words, $\overline{R}$ can be realized as a sequence of one element extensions, each prime over the previous ones–see [1].)

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(5) Let $R \models K_{CR}$. We call R thin if there is a set $\mathcal{P} \subset R_i[x]$, where R_i is the inseparable closure of R–see [1], such that (a) every polynomial $p(x) \in \mathcal{P}$ is normal (in the sense that adjoining one root of $p(x)$ splits $p(x)$ into linear factors) and unambiguous. (b) If $R' \supset R$ splits every polynomial in $\mathcal{P}$ and is generated over R by the roots of these polynomials, then $R' \models K_{\overline{CR}}$. (c) Each $p(x) \in \mathcal{P}$ is defined and monic on some idempotent $e_p(\varepsilon R)$ and $p(x) = p(x)e_p$. (d) If $A \subset \mathcal{P}$ is countable, there is a countable B with $A \subset B \subset \mathcal{P}$ such that if R' results from R by adjoining roots of all the polynomials $p(x) \in B$ (in the sequentially prime way–see [1]), then in $R'[x]$ every polynomial $p(x) \in \mathcal{P}$ factors on e_p into unambiguous monic factors. We shall call such a $\mathcal{P}$ a thin basis for R.

We shall show that if $\overline{R}$ is prime over R, then $\overline{R}$ is sequentially prime over R and consequently that R has a prime integral closure if and only if R is thin.

REMARK. In definition (5) above the only important conditions are (b) and (d); i.e. if we have a set of polynomials which satisfies (b) and (d), then we can construct a set satsifying (a)–(d). Notice also that if R is thin,then $R \models T$.

From now on, when $R \models T$, we shall assume that R is inseparably closed (i.e. every purely inseparable polynomial in $R[x]$ has a root). This involves no loss of generality since the inseparable closure R_i of R always exists and is prime and in fact sequentially prime over R. If R is inseparably closed instead of unambiguous polynomials, we can talk of irreducible polynomials (see [1]). Also all irreducible polynomials are then separable and, consequently, we have the primitive element theorem holding.

Let $R \models T$ and let $\mathcal{P} = \{p(x) \in R[x] \mid p(x)$ is normal, monic and unambiguous$\}$. Let

$$R^* = \prod_{j \in J} R[\{x_p \mid p \in \mathcal{P}\}]_j$$

where the product is over all isomorphism types of $R[\{x_p \mid p \in \mathcal{P}\}]$ such that $p(x_p) = 0$ for all $p \in \mathcal{P}$. Let $\tilde{R}$ be the subring of R^* generated by the sequences $x_p = \{x_{p,j}\}_{j \in J}$, over R. It follows from Lemma 1 of [2] or Lemma 2 of [1] that $\tilde{R}$ is a commutative regular ring. It is not hard to see that $\tilde{R} \models K_{\overline{CR}}$ ($\tilde{R}$ is algebraically closed at each point of $S_{\tilde{R}} = \operatorname{Spec}(\tilde{R})$ and since $S_{\tilde{R}}$ is compact, $\tilde{R}$ is integrally closed). $\tilde{R}$ is a free closure of R in the sense that if $R \subset R_1 \models K_{\overline{CR}}$, then there is a homomorphism of $\tilde{R}$ into R_1 over R–in fact one of the projections will do.

Suppose that R has a prime integral closure $\overline{R}$. Let ν: $\overline{R} \to \tilde{R}$ be a fixed embedding over R. For each $\beta \in \overline{R}$ there is a finite set $X_\beta \subset \{x_p \mid p \in \mathcal{P}\} \subset \tilde{R}$ such that $\nu(\beta) \in R[x_p \mid x_p \in X_\beta]$. If $A \subset \overline{R}$, define $A' \subset \overline{R}$ as follows: $A_0 = A$, $A_{i+1} = \{$all roots in $\overline{R}$ of polynomials $p(x) \in R[x]$ such that $x_p \in \bigcup_{\beta \in A_i} X_\beta\}$ and $A' = \bigcup_{i \in \omega} A_i$. Notice that if $p(x) \in R[x]$, then all the roots of $p(x)$ in $\overline{R}$ are generated by a finite number of roots over R, since $S_{\overline{R}} = S_R$. It follows that if $\overline{\overline{A}} \leqslant \aleph_0$, then $R[A']$ is countably generated over R and, in fact, if $A \subset B$ with $B - A$ countable, then $R[B']$ is countably generated over $R[A']$.

Let $\overline{R} = R[\{x_\alpha \mid \alpha < \lambda\}]$ where each x_α is a root of a polynomial $p(x) \in \mathcal{P}$. Define $A_\alpha = \{x_\gamma \mid \gamma < \alpha\}$. $\overline{R}_\alpha = R[A'_\alpha] \subset \overline{R}$ and $\tilde{R}_\alpha = R[\{x_p \in \tilde{R} \mid a$ is a root of $p(x)$ for some $a \in \overline{R}_\alpha\}] \subset \tilde{R}$.

It is clear that $\overline{R}_\alpha = \nu^{-1}(\tilde{R}_\alpha)$, that $\overline{R}_\delta = \cup_{\alpha<\delta}\overline{R}_\alpha$ for limit ordinals $\delta \leqslant \lambda$, that $\overline{R}_\lambda = \overline{R}$ and that $\overline{R}_{\alpha+1}$ is countably generated over $\overline{R}_\alpha$.

LEMMA 1. (i) *$\overline{R}_\alpha$ is prime over R.*
(ii) *$\overline{R}_{\alpha+1}$ is prime over $\overline{R}_\alpha$.*
(iii) *$\overline{R}_{\alpha+1}$ is sequentially prime over $\overline{R}_\alpha$.*

PROOF. (i) is trivial.
(ii) Since $\tilde{R}_\alpha$ is free over R there is a projection μ: $\tilde{R}_\alpha \to \overline{R}_\alpha$ over R. It is easy to see that $\mu \circ \nu$ is an automorphism of $\overline{R}_\alpha$. Let $\mathcal{I}' = \text{Ker}(\mu) \subset \tilde{R}_\alpha$ and let $\mathcal{I} = \mathcal{I}'\tilde{R}$. Then since $\tilde{R}$ and $\tilde{R}_\alpha$ are models of K_{CR}, $\mathcal{I} \cap \tilde{R}_\alpha = \mathcal{I}'$. Also μ: $\tilde{R}_\alpha/\mathcal{I}' \to \overline{R}_\alpha$ is an isomorphism. It is easy to see that $\tilde{R}/\mathcal{I}$ is free over $\overline{R}_\alpha = \tilde{R}_\alpha/\mathcal{I}'$ (in the same sense that $\tilde{R}$ is free over R). Let $\overline{R}_\alpha \subset R_2 \models K_{\overline{CR}}$. Then there is a homomorphism $\mu_1 : \tilde{R}/\mathcal{I} \to R_2$ over $\overline{R}_\alpha$ so that $\mu_1 \circ \nu$: $\overline{R} \to R_2$ is an embedding. Hence $\overline{R}$ (and thus $\overline{R}_{\alpha+1}$) is prime over R_α. Hence, since $\overline{R}_{\alpha+1}$ is countably generated over R_α, by the remark following Theorem 2 of [1], $\overline{R}_{\alpha+1}$ is sequentially prime over $\overline{R}_\alpha$, and (ii) and (iii) are proved.

COROLLARY. *If $\overline{R}$ is a prime integral closure of R ($\models K_{CR}$), then $\overline{R}$ is sequentially prime over R.*

PROOF. The results of [1] show that $\overline{R} \models T$. The inseparable closure R_i of R is always sequentially prime over R and $R_i \models T$ so the above construction and Lemma 1 show that $\overline{R}$ is sequentially prime over R_i.

LEMMA 2. *If $\overline{R}$ is the prime integral closure of R, then R is thin.*

PROOF. Let $\overline{R} = \cup_{\alpha<\lambda} R_\alpha$ where $\overline{R}_{\alpha+1} = \overline{R}_\alpha[a_\alpha]$ and $p_\alpha(a_\alpha) = 0$ with $p_\alpha(x) \in \overline{R}_\alpha[x]$ irreducible. Let $p_\alpha^*(x) \in R[x]$ be the unique irreducible polynomial in $R[x]$ such that $p_\alpha(x) | p_\alpha^*(x)$. Without loss of generality we may assume that $p_\alpha(x)$ and $p_\alpha^*(x)$ are normal (see the proof of Lemma 1). A set $A \subset \overline{R} - R$ is called downwardly closed if: (i) if $a \in R[A]$ and at some point $z \in S_R$ the first time $a(z)$ occurs in the sequence $\overline{R}_\gamma/z$ is at stage α with $a(z)$ being a combination over R of $a_{\gamma_1}, \ldots, a_{\gamma_n}$ say, then $a_{\gamma_i} \in R[A]$ for $i = 1, \ldots, n$; and (ii) $A = A'$. In the proof of Lemma 1 we saw that if $A = A'$, then $R[A] \models T$, so if A is downwardly closed then $R[A] \models T$. For each downwardly closed $A \subset \overline{R}$ let Y_A^α be a factoring of p_α^* into irreducible factors in $R[A]$. Call two such factorings $Y_{A_1}^\alpha$ and $Y_{A_2}^\alpha$ essentially different if at some point $z \in S_R$ they are different. We now claim that for fixed α there are only finitely many essentially different Y_A^α's (with A downwardly closed). This follows from the fact that p_α^*, factors only finitely often in the well-ordered sequence $\overline{R}_\gamma$ since each $\overline{R}_\gamma \models T$, and that S_R is compact. We leave the details to the reader. For each α choose idempotents $e_{\alpha,i}$, $i = 1, \ldots, n_\alpha$ such that for any downwardly closed A each $p_\alpha^*(x)e_{\alpha,i}$ factors into monic irreducible factors on $e_{\alpha,i}$ for each i. Let

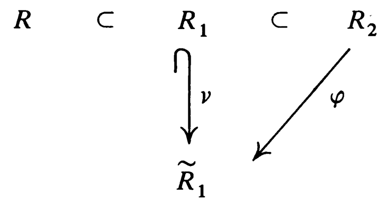
$$\mathcal{P} = \{ p_\alpha^*(x)e_{\alpha,i} | \alpha < \lambda, i = 1, \ldots, n_\alpha \}.$$

Certainly $\mathcal{P}$ satisfies all the conditions of definition (5) except perhaps (d). Let $A \subset \mathcal{P}$ with $\overline{A} = \aleph_0$ and let B_1 be the downward closure of A (defined as follows: For each $a \in R[A] - R$ and each $z \in S_R$, adjoin to A the elements

$a_{\gamma_1}, \ldots, a_{\gamma_n}$ defined above. Since S_R is compact, this will only involve considering a finite number of z's. Call the new set D. Let $A_1 = D'$. Obtain A_{i+1} from A_i in the same way that A_1 was obtained from A. The downward closure of A is $\bigcup_{i<\omega} A_i$). We must show that there is a countable subset $B \subset B_1$ such that adjoining roots for all polynomials in B (in the prime way) causes every polynomial in B_1 to split. Since $\bar{R}$ is separable over R for each $\{a_1, \ldots, a_n\} \subset \bar{R}$ there are essentially only finitely many regular rings between R and $R[a_1, \ldots, a_n]$. By this we mean that there is a finite set of regular rings $R_j, j = 1, \ldots, k$, with $R_j \subset R[a_1, \ldots, a_n]$ and R_j finitely generated over R such that at each point $z \in S_R$, if R_z denotes the field (i.e. stalk) above z, then all the fields between R_z and $R[a_1, \ldots, a_n]_z$ occur among the R_{jz}. For each $\{a_1, \ldots, a_n\} \subset A$ we can look at the rings R_j defined as above and choose a finite set of generators A_j for R_j over R. Let $\bar{A}$ be the union of all these A_j for all finite subsets $\{a_1, \ldots, a_n\}$ of A. Then $\bar{A}$ is countable and in obtaining D from A as above instead of considering all elements of $R[A] - R$ we need only consider all elements of $\bar{A}$. Call this set $\bar{D}$. Let $A_1 = \bar{D}'$ etc. and $B = \bigcup_{i<\omega} A_i$. Then B is countable and downwardly closed. In fact $R[B] = R$ [downward closure of A]. From the definition of $\mathcal{P}$ it is clear that B has the required properties.

Lemma 3. *If $R \subset R_1 \subset R_2$ with $R_1 \models T$ and R_j $(j = 1, 2)$ prime over R then R_2 is prime over R_1.*

Proof. Let $\tilde{R}_1$ be constructed from R_1 as above. We then have



Proof. Let $\mathcal{P}$ be a thin basis for R. Let $A \subset \mathcal{P}$. Then there is a B $(A \subset B \subset \mathcal{P})$ with $\bar{\bar{A}} + \aleph_0 = \bar{\bar{B}} + \aleph_0$ so that every $p \in \mathcal{P}$ factors in R_B (obtained by adjoining roots of polynomials in B) into the product of irreducible monic factors on e_p.

We prove by induction on $\bar{\bar{A}}$ that if $A \subset \mathcal{P}$, then there is a sequentially prime extension R_A of R which splits every polynomial in A and with $R_A \models T$, and such that in $R_A[x]$ every polynomial $p \in \mathcal{P}$ factors into the product of monic irreducible factors on e_p. If A is countable this is trivial. Suppose the assertion is true for all cardinals $< \bar{\bar{A}}$. Let B correspond to A as above. Write $A = \bigcup_{\alpha<\lambda} A_\alpha$ with $A_\delta = \bigcup_{\alpha<\delta} A_\alpha$ for limit ordinals $\delta \leqslant \lambda$, $A_{\alpha+1} \supset A_\alpha$ and $\bar{\bar{A}}_\alpha < \bar{\bar{A}}$ for all $\alpha < \lambda$. Let $B = \bigcup_{\alpha<\lambda} B_\alpha$ with B_α corresponding to A_α as above. Then by induction R_{B_α} exists for each $\alpha < \lambda$. It is clear that $R_{B_\alpha} \models T$ (since in R_{B_α} every polynomial in $\mathcal{P}$ factors into a product of monic irreducible factors) for each $\alpha < \lambda$. Thus by Lemma 3, $R_{B_{\alpha+1}}$ is prime over R_{B_α} and hence $R_B = \bigcup_{\alpha<\lambda} R_{B_\alpha}$ is prime over R.

From Corollary 1 and Lemma 4 we immediately get the

Theorem. *If $R \models K_{CR}$ then R has a prime integral closure if and only if R is thin.*

where φ is an embedding of R_2 into $\tilde{R}_1$ over R which exists because R_2 is prime over R. This diagram need not commute, but we do have $\nu(r) = \varphi(r)$ for $r \in R$. We shall show that there exists an automorphism ψ of $\tilde{R}_1$ over R such that the above diagram with ϕ replaced by $\psi^{-1} \circ \varphi$ does commute. The lemma then follows from the freeness properties of $\tilde{R}_1$ over R_1.

$\tilde{R}_1$ is generated by the x_p, $p \in \mathcal{P}$, over R_1. For $a \in R_1$ let a_i, $i = 1, \ldots, n_a$, denote the conjugates of a over R, and for $p(x) \in \mathcal{P}$ let $p_i(x)$, $i = 1, \ldots, n_p$, denote the conjugates of $p(x)$ over R. Notice that if $p(x) \in \mathcal{P}$, then $p_i(x) \in \mathcal{P}$, and since $R_1 \vDash T$, $(p_i(x), p_j(x)) = 1$ for $i \neq j$.

For $a \in R_1$ we have $\varphi(a) = \sum a_i e_i$ where the e_i are disjoint idempotents of $\tilde{R}_1$ and $\sum e_i = 1$. Similarly we have $\varphi(p(x)) = \sum_{i=1}^{n_p} p_i(x) e_i$.

Define ψ: $\tilde{R}_1 \to \tilde{R}_1$ as follows:

$$\psi(a) = \varphi \circ \nu^{-1}(a) \quad \text{for } a \in \nu(R_1),$$

$$\psi(x_p) = \sum_{i=1}^{n_p} x_{p_i} e_i.$$

It is obvious that ψ is a homomorphism because of the freeness properties of $\tilde{R}_1$ over R_1. ψ is locally one-to-one (i.e. on each stalk) and hence one-to-one. Also $x_p \in$ Range (ψ) so ψ is onto. Therefore ψ is an automorphism with the required properties.

LEMMA 4. *If R is thin, then R has a prime integral closure.*

REMARK. The condition that R be thin is not a first order condition since every countable model of T is thin. Hence for R uncountable the necessary and sufficient condition for R to have a prime integral closure is not first order, while for countable R it is.

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