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DIOPHANTINE INEQUALITIES WITH MIXED POWERS (mod 1)

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ABSTRACT. A theorem of Heilbronn on the distribution of the sequence $n^2\theta$ (mod 1) is extended to sums of mixed powers.

1. **Introduction.** In 1948 Heilbronn [6] proved that for any $\varepsilon > 0$ there exists $C(\varepsilon)$ such that for any real θ and $N \geqslant 1$ there is an integer x satisfying

$$1 \leqslant x \leqslant N \quad \text{and} \quad \|\theta x^2\| < C(\varepsilon) N^{-1/2+\varepsilon}, \tag{1}$$

where $\|\alpha\|$ denotes the difference between α and the nearest integer, taken positively. The result has been extended in several directions. In particular, in [2] an analogous result was obtained for the fractional parts of an additive form

$$\theta_1 x_1^k + \cdots + \theta_s x_s^k \tag{2}$$

of degree k. Here we shall prove analogous results for mixed powers.

THEOREM 1. *Let $2 \leqslant k_1 \leqslant \cdots \leqslant k_s$ be integers, $K_i = 2^{k_i - 1}$ and*

$$l = K_1^{-1} + \cdots + K_s^{-1}. \tag{3}$$

If $l \geqslant 1$ then for any $\varepsilon > 0$, $N > 1$ and real $\theta_1, \ldots, \theta_s$ there exist integers $x_1, \ldots, x_s$ such that

$$0 < \max|x_i| \leqslant N \quad \textit{and} \quad \|\theta_1 x_1^{k_1} + \cdots + \theta_s x_s^{k_s}\| < CN^{-1+\varepsilon}, \tag{4}$$

where C depends only on $k_1, \ldots, k_s$ and ε.

EXAMPLE 1. *Let $k_1 = 2$, $k_2 = k_3 = 3$. For any $\varepsilon > 0$ there is a constant $C = C(\varepsilon)$ such that for any $N > 1$ and real $\theta_1, \theta_2, \theta_3$ there are integers x_1, x_2, x_3 satisfying*

$$0 < \max|x_i| \leqslant N \quad \textit{and} \quad \|\theta_1 x_1^2 + \theta_2 x_2^3 + \theta_3 x_3^3\| < CN^{-1+\varepsilon}.$$

THEOREM 2. *Let $2 \leqslant k_1 \leqslant \cdots \leqslant k_s$ be integers and let l be defined by* (3). *If $l < 1$ then for any $\varepsilon > 0$, $N > 1$ and real $\theta_1, \ldots, \theta_s$ there exist integers $x_1, \ldots, x_s$ satisfying*

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(5) $0 < \max|x_i| \leqslant N$ *and* $\|\theta_1 x_1^{k_1} + \cdots + \theta_s x_s^{k_s}\| < CN^{-s/(s+(1-l)K_s)+\varepsilon}$,

where C depends only on $k_1, \ldots, k_s$ and ε.

EXAMPLE 2. *Let $k_1 = 2$, $k_2 = 3$. For any $\varepsilon > 0$ there is a constant $C = C(\varepsilon)$ such that for any $N > 1$ and real θ_1, θ_2 there are integers x_1, x_2 satisfying*

$$0 < \max|x_i| \leqslant N \quad \textit{and} \quad \|\theta_1 x_1^2 + \theta_2 x_2^3\| < CN^{-2/3+\varepsilon}.$$

EXAMPLE 3. *Let $s = k - 1$, $k_1 = 2$, $k_2 = 3, \ldots, k_s = k$. For any $\varepsilon > 0$ there is a constant $C = C(\varepsilon)$ such that for any $N > 1$ and real $\theta_1, \ldots, \theta_{k-1}$ there are integers $x_1, \ldots, x_{k-1}$ satisfying*

$$0 < \max|x_i| \leqslant N \quad \textit{and} \quad \|\theta_1 x_1^2 + \theta_2 x_2^3 + \cdots + \theta_{k-1} x_{k-1}^k\| < CN^{-1+1/k+\varepsilon}.$$

Recently Chong and Liu [1] obtained an analogue of Heilbronn's theorem for the sum $P_1(x_1) + \cdots + P_s(x_s)$ of s polynomials, having no constant term, where each of the polynomials is of exact degree k. They proved that if $s \geqslant K$ then the inequalities

$$0 < \max|x_i| \leqslant N, \qquad \|P_1(x_1) + \cdots + P_s(x_s)\| \ll N^{-1/K+\varepsilon} \tag{6}$$

have a solution. However, if one of the polynomials, P_1 say, is of degree at most $k - 1$ then Davenport [3] has shown that there is an integer x satisfying

$$1 \leqslant x \leqslant N \text{ and } \|P_1(x)\| \ll N^{-1/(K+1)+\varepsilon}.$$

It therefore appears that an improved estimate for the sum of polynomials with differing degrees would require an improvement in the estimate when the degrees are all equal.

2. **Preliminary lemmas and notation.** We may suppose that ε is a small positive number; let η be a small positive number which can be chosen as an explicit function of ε. We may suppose that $N > N_0(k_1, \ldots, k_s, \varepsilon)$. By $F \ll G$ we mean that $|F| < CG$ where C depends at most on $k_1, \ldots, k_s$ and ε. We write $e(z)$ for $\exp(2\pi iz)$.

LEMMA 1 (VINOGRADOV). *Let Δ satisfy $0 < \Delta < \frac{1}{2}$ and let a be a positive integer. There exists a function $\psi(z)$, periodic with period 1, which satisfies*

$$\psi(z) = 0 \quad \textit{for } \|z\| \geqslant \Delta \tag{7}$$

and

$$\psi(z) = \sum_{v=-\infty}^{\infty} a_v e(vz) \tag{8}$$

where the a_v are real numbers, $a_0 = \Delta$, $a_{-v} = a_v$ and

$$|a_v| < A \min(\Delta, v^{-a-1}\Delta^{-a}) \qquad (v \neq 0), \tag{9}$$

where A depends only on a.

This is a particular case of Lemma 12 of Chapter 1 of Vinogradov [7].

LEMMA 2 (DIRICHLET). *Let θ be a real number and $Q \geqslant 1$. Then there exists an integer q satisfying*

$$1 \leqslant q \leqslant Q \quad \textit{and} \quad \|q\theta\| \leqslant Q^{-1}. \tag{10}$$

See, for example, Theorem 36 of Hardy and Wright [5].

LEMMA 3 (WEYL). *Let $k \geqslant 2$ be an integer, $K = 2^{k-1}$ and*

$$P(x) = \theta x^k + \theta_{k-1} x^{k-1} + \cdots + \theta_1 x + \theta_0 \tag{11}$$

be a polynomial of exact degree k, i.e. $\theta \neq 0$. Let B, N be real numbers with $N \geqslant 1$. Then for any $\eta > 0$,

$$\Big|\sum_{B \leqslant x \leqslant B+N} e(P(x))\Big|^K \ll N^\eta\Big(N^{K-1} + N^{K-k} \sum_{m=1}^{L} \min(N, \|m\theta\|^{-1})\Big) \tag{12}$$

where $L = k!\,N^{k-1}$.

See, for example, Lemma 1 of Davenport [4].

3. **Proof of Theorem** 1. Suppose that $K_1^{-1} + \cdots + K_j^{-1} < 1 \leqslant K_1^{-1} + \cdots + K_j^{-1} + K_{j+1}^{-1}$. Since $K_1^{-1} + \cdots + K_j^{-1}$ is a rational number whose denominator divides K_j, and K_{j+1} is a multiple of K_j, it follows that $K_1^{-1} + \cdots + K_j^{-1} + K_{j+1}^{-1} = 1$. Therefore it is sufficient to prove Theorem 1 when $l = 1$, since the general result follows on taking $x_{j+2} = x_{j+3} = \cdots = x_s = 0$.

Let $M = N^{1-\varepsilon}$ and suppose that $\|\theta_1 x_1^{k_1} + \cdots + \theta_s x_s^{k_s}\| \geqslant M^{-1}$ for $1 \leqslant x_i \leqslant N$, $i = 1, \ldots, s$. We apply Lemma 1 with $\Delta = M^{-1}$ and obtain

$$\sum_{x_1=1}^{N} \cdots \sum_{x_s=1}^{N} \psi(\theta_1 x_1^{k_1} + \cdots + \theta_s x_s^{k_s}) = 0.$$

Therefore

$$M^{-1}N^s + \sum_{v \neq 0} a_v \prod_{i=1}^{s} S_v(i) = 0 \quad \text{where } S_v(i) = \sum_{x=1}^{N} e(v\theta_i x^{k_i}).$$

Then

$$M^{-1}N^s \ll \sum_{v=1}^{\infty} |a_v| \prod_{i=1}^{s} |S_v(i)|. \tag{13}$$

Let $M_1 = M^{1+\eta}$; then the contribution of those $v > M_1$ is $\ll N^{s-a\eta}$, and taking $a = [3\eta^{-1}]$ we obtain

$$M^{-1}N^s \ll \sum_{v=1}^{M_1} |a_v| \prod_{i=1}^{s} |S_v(i)| \ll M^{-1} \sum_{v=1}^{M_1} \prod_{i=1}^{s} |S_v(i)|.$$

Since $l = 1$ an application of Hölder's inequality gives

$$N^s \ll \prod_{i=1}^{s} \Big\{\sum_{v=1}^{M_1} |S_v(i)|^{K_i}\Big\}^{1/K_i}. \tag{14}$$

Therefore for some value i, for which we write $S_v(i)$, θ_i, k_i, K_i as S_v, θ, k and K respectively, we have $N^K \ll \sum_{v=1}^{M_1} |S_v|^K$. Applying Lemma 3 we have

$$N^K \ll \sum_{v=1}^{M_1} N^\eta \Big(N^{K-1} + N^{K-k} \sum_{m=1}^{L} \min(N, \|vm\theta\|^{-1}) \Big). \tag{15}$$

The contribution of the first term on the right-hand side is $M_1 N^\eta N^{K-1} \ll N^{K-\varepsilon+2\eta}$ which is $o(N^K)$ provided that $\eta < \varepsilon/2$. Therefore

$$N^K \ll N^{K-k+\eta} \sum_{v=1}^{M_1} \sum_{m=1}^{L} \min(N, \|vm\theta\|^{-1}) \ll N^{K-k+2\eta} \sum_{h=1}^{H} \min(N, \|h\theta\|^{-1})$$

where $h = mv$, $H = M_1 L$ and we have used the fact that the number of representations of h in the form mv is $\ll N^\eta$. Thus

$$N^{k-2\eta} \ll \sum_{h=1}^{H} \min(N, \|h\theta\|^{-1}). \tag{16}$$

We take $Q = N^{k-3\eta}$ and choose an integer q satisfying (10). Rearranging the sum on the right-hand side of (16) into blocks of length q and estimating in the usual way, see, for example, [4, p. 13], we obtain

$$N^{k-2\eta} \ll (q^{-1}H + 1)(N + q \log q). \tag{17}$$

Clearly $N = o(N^{k-2\eta})$,

$$q \log q \ll QN^{\eta/2} = o(N^{k-2\eta}) \quad \text{and} \quad H \log q \ll M_1 N^{k-1+\eta} \ll N^{k-\varepsilon+2\eta}$$

which is $o(N^{k-2\eta})$ provided that $\eta < \varepsilon/4$.

Therefore

$$N^{k-2\eta} \ll q^{-1} HN \tag{18}$$

so that

$$q \ll M_1 N^{2\eta}. \tag{19}$$

Then, for $N > N_0(k_1, \ldots, k_s, \varepsilon)$ we have $1 \leqslant q \leqslant N$ and

$$\|q^k\theta\| \leqslant q^{k-1}\|q\theta\| \leqslant q^{k-1}Q^{-1} \ll M_1^{k-1} N^{-k+(2k+1)\eta} < N^{-1} \tag{20}$$

provided that η is sufficiently small. Thus $x_i = q$, $x_j = 0$ when $j \neq i$ gives a solution of the inequalities (4), which completes the proof of Theorem 1.

4. **Proof of Theorem** 2. If the inequalities (5) have no solution we take $t = s/(s + K_s(1 - l))$, $M = N^{t-\varepsilon}$ and proceed as for Theorem 1. In place of (14) we obtain

$$N^s \ll M_1^{1-l} \prod_{i=1}^{s} \Big\{ \sum_{v=1}^{M_1} |S_v(i)|^{K_i} \Big\}^{1/K_i}. \tag{21}$$

Thus for some i, for which we write θ_i as θ, etc.,

$$M_1^{l-1} N^s \ll \Big\{ \sum_{v=1}^{M_1} |S_v|^K \Big\}^{s/K}. \tag{22}$$

Applying Weyl's estimate we have

$$(23)\qquad \begin{aligned} M_1^{K(l-1)/s} N^K &\ll \sum_{v=1}^{M_1} |S_v|^K \\ &\ll \sum_{v=1}^{M_1} N^\eta \Big(N^{K-1} + N^{K-k} \sum_{m=1}^{L} \min(N, \|vm\theta\|^{-1}) \Big). \end{aligned}$$

Suppose first that

$$(24)\qquad M_1^{K(l-1)/s} N^K \ll M_1 N^\eta N^{K-1},$$

then

$$N^{1-\eta} \ll M_1^{1+K(1-l)/s} \ll M_1^{1+K_s(1-l)/s} = M_1^{1/t}.$$

Therefore $1 - \varepsilon/t \geqslant (1-\eta)/(1+\eta)$ and choosing η sufficiently small we obtain a contradiction. Therefore

$$M_1^{k(l-1)/s} N^K \ll N^{K-k+2\eta} \sum_{h=1}^{H} \min(N, \|h\theta\|^{-1})$$

where $h = mv$ and $H = M_1 L$. Hence

$$(25)\qquad M_1^{K(l-1)/s} N^{k-2\eta} \ll \sum_{h=1}^{H} \min(N, \|h\theta\|^{-1}).$$

We take $Q = M_1^{K(l-1)/s} N^{k-3\eta}$. Since

$$(26)\qquad k - K(1-l)t/s = k - K(1-l)/(s + K_s(1-l)) \geqslant k - 1$$

we have $Q \geqslant 1$ and so there exists an integer q satisfying (10). Rearranging the sum on the right-hand side of (25) into blocks of q terms we obtain

$$(27)\qquad M_1^{K(l-1)/s} N^{k-2\eta} \ll (q^{-1}H + 1)(N + q \log q).$$

Using (26) we have $N = o(M_1^{K(l-1)/s} N^{k-2\eta})$ and

$$q \log q \ll Q N^{\eta/2} = o(M_1^{K(l-1)/s} N^{k-2\eta}).$$

Suppose that

$$(28)\qquad M_1^{K(l-1)/s} N^{k-2\eta} \ll H \log q \ll M_1 N^{k-1+\eta};$$

then

$$N^{1-3\eta} \ll M_1^{1+K(1-l)/s} \ll M_1^{1/t}.$$

Therefore $1 - 3\eta \leqslant (1+\eta)(1-\varepsilon/t)$ which gives a contradiction, provided that η is sufficiently small. Therefore (28) is false and so

$$M_1^{K(l-1)/s} N^{k-2\eta} \ll q^{-1} HN$$

which implies

$$q \ll N^{2\eta} M_1^{1+K(1-l)/s} \ll N^{2\eta} M_1^{1/t}. \tag{29}$$

Then $1 \leqslant q \leqslant N$ and

$$\begin{aligned}\|q^k\theta\| &\leqslant q^{k-1}\|q\theta\| \leqslant q^{k-1}Q^{-1} \\ &\ll N^{-k+(2k+1)\eta} M_1^{(k-1)/t+K(1-l)/s} \ll N^{-t+\varepsilon}\end{aligned}$$

since

$$-k + t\left\{\frac{k-1}{t} + \frac{K(1-l)}{s}\right\} \leqslant -1 + \frac{tK_s(1-l)}{s} = -t,$$

and this completes the proof of Theorem 2.

References

1. K.-K. Chong and M. C. Liu, *The fractional parts of a sum of polynomials*, Monatsh. Math. (to appear).
2. R. J. Cook, *The fractional parts of an additive form*, Proc. Cambridge Philos. Soc. **72** (1972), 209–212. MR **45** #6764.
3. H. Davenport, *On a theorem of Heilbronn*, Quart. J. Math. Oxford Ser. (2) **18** (1967), 339–344. MR **36** #6355.
4. ———, *Analytic methods for Diophantine equations and Diophantine inequalities*, Ann Arbor, Ann Arbor, Mich., 1963. MR **28** #3002.
5. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4th ed., Oxford Univ. Press, Oxford, 1965.
6. H. Heilbronn, *On the distribution of the sequence $n^2\theta$* (mod 1), Quart. J. Math. Oxford Ser. **19** (1948), 249–256. MR **10**, 284.
7. I. M. Vinogradov, *The method of trigonometrical sums in the theory of numbers*, Trudy Mat. Inst. Steklov. **23** (1947); English transl., Interscience, New York and London, 1954. MR **10**, 599; **15**, 941.

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