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STRONGLY STRICTLY CYCLIC WEIGHTED SHIFTS

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ABSTRACT. An example is given of a unilateral weighted shift on complex Hilbert space which is strictly cyclic but not strongly strictly cyclic. Similar examples of weighted shifts on the sequence spaces l^p, for $1 < p < \infty$, are indicated.

1. **Introduction.** A unilateral weighted shift operator T on complex separable Hilbert space H is an operator that maps each vector in some orthonormal basis $\{e_n\}_0^\infty$ into a scalar multiple of the next basis vector: $Te_n = w_n e_{n+1}$ for each n. The shift T is said to be strictly cyclic if there exists a vector $x \in H$ such that $A(T)x = H$, where $A(T)$ is the weakly closed algebra generated by T. Let E_n denote the span in H of the basis vectors $\{e_i : i \geqslant n\}$. If the restricted operators $T|E_n$ are all strictly cyclic (for $n = 0, 1, 2, \ldots$), then T is said to be strongly strictly cyclic. A. L. Shields, in his survey of weighted shifts, asked if every strictly cyclic weighted shift is strongly strictly cyclic [3, Question 14]. This note gives an example which shows that, even under the additional hypothesis that the weight sequence of the operator is monotonically decreasing to 0, a strictly cyclic operator T may have a restriction $T|E_1$ which fails to be strictly cyclic.

2. **Statement of results.** The notation of [3] is followed. Let T denote a unilateral weighted shift on H with a weight sequence w_n such that $w_0 \geqslant w_1 \geqslant w_2 \geqslant \cdots > 0$. Define $\beta(0) = 1$ and $\beta(n) = w_0 w_1 \cdots w_{n-1}$ for $n \geqslant 1$, so that $w_n = \beta(n+1)/\beta(n)$. Let $\beta(n,j) = (\beta(n)/\beta(j)\beta(n-j))^2$, for $0 \leqslant j \leqslant n$, and let $S_\beta(n) = \sum_{j=0}^n \beta(n,j)$. In order to determine if T is strictly cyclic it suffices, by the following result, to examine the sequence $\{S_\beta(n)\}$.

THEOREM [2, THEOREM 3.2], [3, PROPOSITION 32]. *If $\{w_n\}$ is monotonically decreasing, then T is strictly cyclic if and only if the sequence $\{S_\beta(n)\}$ is bounded.*

COROLLARY. *If $\{w_n\}$ is monotonically decreasing to* 0 *and if* $2 < \limsup S_\beta(n) < \infty$, *then T is strictly cyclic but $T|E_1$ is not. Thus T is strictly cyclic but not strongly strictly cyclic.*

PROOF. It is no loss of generality to assume that $w_0 = 1$: if $w_0 \neq 1$, multiply each weight by $(w_0)^{-1}$ and observe that this change does not affect the values of $\beta(n,j)$ or $S_\beta(n)$. The operator $T|E_1$ is itself a weighted shift, with weights $v_n = w_{n+1}$. Define $\alpha(n)$ for $\{v_n\}$ as $\beta(n)$ was defined for $\{w_n\}$, so that $\alpha(0) = 1 = \beta(1)$ and $\alpha(n) = v_0 v_1 \cdots v_{n-1} = \beta(n+1)$. Thus

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$$\begin{aligned}\alpha(n,j) &= (\alpha(n)/\alpha(j)\alpha(n-j))^2 \\ &= (\beta(n+1)/\beta(j+1)\beta(n+1-j))^2.\end{aligned}$$

Hence

$$S_\alpha(n) = \sum_{j=0}^{n} \alpha(n,j) = \sum_{k=1}^{n+1} \left(\frac{\beta(n+1)}{\beta(k)\beta(n+2-k)}\right)^2$$
$$+\left(\frac{\beta(n+1)}{\beta(n+2)}\right)^2 \sum_{k=1}^{n+1} \beta(n+2,k) = (w_{n+1})^{-2}[S_\beta(n+2) - 2]$$

since $\beta(n+2,0) = \beta(n+2,n+2) = 1$. If $w_n \downarrow 0$ and if $2 < \limsup S_\beta(n) < \infty$, it follows that $\{S_\alpha(n)\}$ is unbounded, so that $T|E_1$ is not strictly cyclic.

3. **Construction.** To find weights $\{w_n\}$ which satisfy the conditions of the Corollary, proceed as follows. Note first that, for $n \geqslant 2$, $S_\beta(n) = 2 + (w_{n-1})^2 K_n$, where K_n is a function of $w_0, \ldots, w_{n-2}$ only. It follows that $S_\beta(n)$ is an increasing function of w_{n-1}. Consequently, if $w_0, \ldots, w_{n-2}$ are already chosen, w_{n-1} may then be chosen to make $S_\beta(n)$ take on any value larger than 2. The trick is to achieve large values of $S_\beta(n)$ with a sequence $\{w_n\}$ which is monotonically decreasing to 0.

Take any value $L > 2$, and then choose the w_n inductively as follows. Let $w_0 = w_1 = 1$, and suppose that, for $n \geqslant 3$, the weights $w_0, \ldots, w_{n-2}$ have been chosen so that $S_\beta(n-1) \leqslant L$. Let $w_{n-1} = w_{n-2}$ if this choice will make $S_\beta(n) \leqslant L$ (Case 1); otherwise let w_{n-1} be the smaller of $\frac{1}{2}w_{n-2}$ (Case 2a) and that value which would make $S_\beta(n) = L$ (Case 2b). In either case $S_\beta(n) \leqslant L$ and $w_{n-1} \leqslant w_{n-2}$.

For integers n of Case 2, $S_\beta(n) \geqslant 3/2 + L/4 > 2$. For, if Case 2a occurs, then $2 + (w_{n-2})^2 K_n > L$. Hence

$$\begin{aligned}S_\beta(n) &= 2 + (w_{n-2})^2 K_n/4 \\ &= 3/2 + [2 + (w_{n-2})^2 K_n]/4 > 3/2 + L/4.\end{aligned}$$

In Case 2b, $S_\beta(n) = L > 3/2 + L/4$. Thus $\limsup S_\beta(n) > 2$ if Case 2 occurs infinitely often. This must be true, for otherwise the sequence $\{w_n\}$ is eventually constant. An easy computation then shows that $\lim S_\beta(n) = +\infty$, contradicting the fact that $S_\beta(n) \leqslant L$. Since $w_{n-1} \leqslant \frac{1}{2}w_{n-2}$ for integers n of Case 2 it follows that the sequence $\{w_n\}$ decreases monotonically to 0.

4. **Remarks.** The construction above may easily be modified to make $S_\beta(n)$ unbounded. This yields an example of a weighted shift whose weights decrease monotonically to 0 and yet which is not strictly cyclic. (A similar example is given by D. A. Herrero [1].)

Weighted shifts may be defined on the sequence spaces l^p. The Theorem of Kerlin and Lambert stated above holds for operators on l^p with $1 < p < \infty$ if it is modified so that $\beta(n,j)$ is defined by $\beta(n,j) = (\beta(n)/\beta(j)\beta(n-j))^q$, where $q = p/(p-1)$. (Kerlin and Lambert prove the Theorem in this general form.) The construction above is easily modified to fit this situation. Thus a strictly cyclic weighted shift on l^p, with $1 < p < \infty$, need not be strictly cyclic, even though its weight sequence decreases monotonically to 0.

References

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