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PARTITIONS WITH CONGRUENCE CONDITIONS

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ABSTRACT. Let $A = \bigcup_{i=1}^{q}\{a(i) + \nu M: \nu = 0,1,2,\dots\}$, where q, M and the $a(i)$ are positive integers such that $a(1) < a(2) < \cdots < a(q) \leqslant M$. Asymptotic formulae are obtained for $p(n,k,A)$, $p^*(n,k,A)$ the number of partitions of n into k parts, k unequal parts respectively, where all the parts belong to A.

Since the celebrated Hardy-Ramanujan paper [4] which gave an asymptotic formula for $p(n)$, the number of unrestricted partitions of the positive integer n, many authors have evaluated different cases of $p(n,A)$, the number of partitions of n into parts belonging to

$$A = \bigcup_{i=1}^{q}\{a(i) + \nu M: \nu = 0,1,2,\dots\},$$

where q, M and the $a(i)$ are positive integers such that $a(1) < a(2) < \cdots < a(q) \leqslant M$. The Hardy-Ramanujan circle method as modified by Rademacher [9] was employed to obtain convergent series expressions for $p(n,A)$. When $q = 2$, $a(2) = M - a(1)$, Niven [8] investigated the case $M = 6$, Lehner [6] the case $M = 5$ and Livingood [7] the case where M is any prime > 3. Later Iseki [5] evaluated $p(n,A)$ when M is any composite > 3 and $a(1)$, M are relatively prime. Then Hagis [3] considered the case of all odd primes M, where $q = 2r$ and $a(h) + a(2r - h) = M$ for $1 \leqslant h \leqslant r$.

In all these cases, A is symmetric in the sense that $a(h) \in A$ implies that $M - a(h) \in A$ and this ensures that the generating function of the $p(n,A)$ is a modular form. Rademacher's method then leads to a convergent series representation of $p(n,A)$. Grosswald [2] investigated the case where M is any odd prime and A is an arbitrary asymmetrical set. Then the above method cannot be applied and only asymptotic results are obtained.

In [1], Erdös and Lehner by means of more elementary methods investigated $p(n,k)$, the number of partitions of n into k parts, and proved that $p(n,k) \sim n^{k-1}/k!\,(k-1)!$ as $n \to \infty$, provided that $k = o(n^{1/3})$. This formula does not appear to have been generalized in the above manner and it is the object of this note to do so.

We denote by $p(n,k,A)$, $p^*(n,k,A)$ the number of partitions of n into k parts, k unequal parts respectively, where all the parts belong to A. For every

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positive integer n, we define r by $r \equiv n \pmod{M}$, $1 \leqslant r \leqslant n$. We write

$$(1) \qquad P(x,k,A) = \left\{ \sum_{i=1}^{q} x^{a(i)} \right\}^k = \sum_{\nu=1}^{kM} c_\nu x^\nu,$$

$$(2) \qquad S_r\{P(x,k,A)\} = \sum_{\nu=0}^{k-1} c_{\nu M+r}$$

and d, δ for the greatest common divisors $(a(1), \ldots, a(q), M)$, $(a(2) - a(1), \ldots, a(q) - a(1), M)$. $\{\delta = M$ for $q = 1.\}$ Clearly $d = (a(1), \delta)$. We can now state our result.

THEOREM. *For any given k, r for which $S_r\{P(x,k,A)\} \neq 0$,*

$$p(Mn + r, k, A) \sim p^*(Mn + r, k, A) \sim n^{k-1} S_r\{P(x,k,A)\}/k!\,(k-1)!$$

as $n \to \infty$. If δ divides r, $n \to \infty$ and $k \to \infty$ through multiples of δ/d subject to the condition that $k = o(n^{1/4})$, then

$$p(Mn + r, k, A) \sim p^*(Mn + r, k, A) \sim n^{k-1} q^k \delta / M\{k!\,(k-1)!\}.$$

We observe that the second part of our theorem is established only for $k = o(n^{1/4})$ instead of under the condition $k = o(n^{1/3})$ of Erdös and Lehner. The proof here is, in fact, quite different from that of [1] where the condition on k is obtained by using the fact that the number of partitions of n into k unequal parts is $p(n - \frac{1}{2}k(k-1), k)$. This appears to have no obvious generalization for partitions into parts belonging to A.

PROOF OF THE THEOREM. We write

$$G(y) = G(y,x,A) = \prod_{\nu=0}^{\infty} \prod_{i=1}^{q} \{1 - x^{\nu M + a(i)} y\}^{-1}$$
$$= 1 + \sum_{k=1}^{\infty} g(k) y^k,$$

where $g(k) = g(k,x,A)$ is the generating function of $p(n,k,A)$. Therefore,

$$\log G(y) = - \sum_{\nu=0}^{\infty} \sum_{i=1}^{q} \log\{1 - x^{\nu M + a(i)} y\}$$
$$= \sum_{\nu=0}^{\infty} \sum_{i=1}^{q} \sum_{h=1}^{\infty} h^{-1} x^{h\{\nu M + a(i)\}} y^h$$
$$= \sum_{h=1}^{\infty} h^{-1} y^h \beta(h),$$

where

$$(3) \qquad \beta(h) = \beta(h,x,A) = (1 - x^{hM})^{-1} \sum_{i=1}^{q} x^{ha(i)}.$$

Hence

$$G(y) = \exp\left\{\sum_{h=1}^{\infty} h^{-1}y^h\beta(h)\right\}$$

and so,

$$g(k) = \sum_{(k)} \prod_{m} \{h(m)!\}^{-1}\{m^{-1}\beta(m)\}^{h(m)}, \tag{4}$$

where the sum is taken over all unrestricted partitions of k of the form $k = \sum_{m=1}^{k} h(m)m$ and the product is taken over all the different parts m of the partition.

The term on the right-hand side of (4) corresponding to the partition of k into k units is

$$\begin{aligned}(k!)^{-1}(1 - x^M)^{-k}\left\{\sum_{i=1}^{q} x^{a(i)}\right\}^k \\ = (k!)^{-1}(1 - x^M)^{-k}\sum_{\nu=1}^{kM} c_\nu x^\nu\end{aligned}$$

by (1), and the coefficient of x^{Mn+r} in this term is

$$(k!)^{-1}\sum_{\nu=0}^{k-1} c_{\nu M+r}\binom{n-\nu+k-1}{k-1}. \tag{5}$$

It follows that, as $n \to \infty$, this coefficient

$$\begin{aligned}&\sim n^{k-1}\sum_{\nu=0}^{k-1} c_{\nu M+r}/k!\,(k-1)! \\ &= n^{k-1}S_r\left\{P(x,k,A)\right\}/k!\,(k-1)!\end{aligned}$$

by (2).

From (3), the general term on the right-hand side of (4) is

$$\prod_{m} \{h(m)!\}^{-1}m^{-h(m)}(1 - x^{mM})^{-h(m)}\left\{\sum_{i=1}^{q} x^{ma(i)}\right\}^{h(m)}.$$

Since, for all $\nu > 0$, the coefficient of x^ν in $\sum_{i=1}^{q} x^{ma(i)}$ is not greater than the coefficient of x^ν in $\{\sum_{i=1}^{q} x^{a(i)}\}^m$, the coefficient of x^{Mn+r} in this general term is less than

$$Cn^{-1}\prod_{m} nm^{-h(m)}(n/m)^{h(m)-1}S_r\{P(k,A)\}/h(m)!\,\{h(m)-1\}!,$$

where C is independent of n, k. Hence, in order to show that

$$p(Mn+r,k,A) \sim n^{k-1}S_r\{P(x,k,A)\}/k!\,(k-1)!$$

as $n \to \infty$, we must prove that

$$n^{-k}k!\,(k-1)!\sum_{(k)}{}' \prod_{m} n^{h(m)}m^{1-2h(m)}/h(m)!\,\{h(m)-1\}! = o(1) \tag{6}$$

as $n \to \infty$, where the sum is taken over all partitions $\sum h(m)m$ of k into less than k parts.

Now, since any partition of k into $k - \mu$ parts, where $\mu < \frac{1}{2}k$, must contain at least $k - 2\mu$ units, we have

$$\prod_m h(m)!\,\{h(m) - 1\}! \geqslant \Lambda(k - 2\mu)$$

where $\Lambda(k - 2\mu) = (k - 2\mu)!\,(k - 2\mu - 1)!$ for $\mu < \frac{1}{2}k$ and $\Lambda(k - 2\mu) = 1$ for $\mu \geqslant \frac{1}{2}k$. Also, the number of partitions of k into $k - \mu$ parts is less than or equal to $p(\mu)$, the number of unrestricted partitions of μ, according as $\mu > \frac{1}{2}k$ or $\mu \leqslant \frac{1}{2}k$. Therefore, since for all $\mu > 0$, $p(\mu) < \exp\{\pi\surd(2\mu/3)\}$, the left-hand side of (6) is less than

$$\begin{aligned}
&\sum_{\mu=1}^{k-1} \exp\{\pi\surd(2\mu/3)\}k^{4\mu}n^{-\mu} \\
&\qquad < \sum_{\mu=1}^{k-1} \exp\{\pi\surd(2\mu/3) + 4\mu \log k - \mu \log n\} \\
&\qquad < C' \sum_{\mu=1}^{\infty} \exp\{-\tfrac{1}{2}\mu \log(n/k^4)\},
\end{aligned}$$

where C' is independent of n, k. (6) follows immediately.

If we write

$$\begin{aligned}
G^*(y) = G^*(y, x, A) &= \prod_{\nu=0}^{\infty} \prod_{i=1}^{q} \{1 + x^{\nu M + a(i)}y\} \\
&= 1 + \sum_{k=1}^{\infty} g^*(k)y^k,
\end{aligned}$$

then it follows that

$$\log G^*(y) = \sum_{h=1}^{\infty} (-1)^{h-1}h^{-1}y^h\beta(h)$$

and

$$g^*(k) = \sum_{(k)} \prod_m \{h(m)!\}^{-1}\{(-1)^{m-1}m^{-1}\beta(m)\}^{h(m)}. \tag{7}$$

The proof of the asymptotic formula for $p^*(Mn + r, k, A)$ then follows from (7) exactly as the proof of the formula for $p(Mn + r, k, A)$ followed from (4).

In order to prove the second part of the theorem, we let ω be a primitive Mth root of unity. Then, from (1) and (2),

$$P(\omega^h, k, A) = \sum_{\nu=1}^{kM} c_\nu \omega^{\nu h} = \sum_{r=1}^{M} S_r \omega^{rh}$$

for $1 \leqslant h \leqslant M$, where $S_r = S_r\{P(x, k, A)\}$. It is a simple application of alternant theory to solve these equations and obtain

$$S_r = M^{-1} \sum_{\nu=0}^{M-1} \omega^{-r\nu} P(\omega^\nu, k, A) \tag{8}$$

for $1 \leqslant r \leqslant M$. From the definition of δ, $\sum_{i=1}^{q} \omega^{\nu a(i)} = q\omega^{\nu a(1)}$ whenever M/δ divides ν and $|\sum_{i=1}^{q} \omega^{\nu a(i)}| < q$ otherwise. Hence, since δ divides r and $(a(1), \delta) = d$, we see from (1) and (8) that $S_r \sim q^k \delta/M$ as $k \to \infty$ through multiples of δ/d. The second part of the theorem now follows from (5) by letting n, $k \to \infty$, since (6) follows as before for $k = o(n^{1/4})$.

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