PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 57, Number 1, May 1976

A GENERALIZATION OF THE WIENER-LÉVY THEOREM APPLICABLE TO SOME VOLTERRA EQUATIONS

G. S. JORDAN AND ROBERT L. WHEELER

ABSTRACT. Recently, Shea and Wainger obtained a variant of the Wiener-Lévy theorem for nonintegrable functions of the form $a(t) = b(t) + \beta(t)$, where $b(t)$ is nonnegative, nonincreasing, convex and locally integrable, and $\beta(t), t\beta(t) \in L^1(0, \infty)$. It is shown here that the moment condition $t\beta(t) \in L^1$ may be omitted from the hypotheses of this theorem. These results are useful in the study of stability problems for some Volterra integral and integrodifferential equations.

Introduction. It is well known that under rather weak hypotheses (see [9, Chapter 4] and [3]) the solutions of the linear Volterra equations

$$u(t) = f(t) - \int_0^t a(t-s)u(s)\,ds \qquad (0 \leqslant t < \infty), \tag{1}$$

$$u'(t) = f(t) - \int_0^t a(t-s)u(s)\,ds \qquad (u(0) = u_0; 0 \leqslant t < \infty) \tag{2}$$

may be written as

$$u(t) = f(t) - \int_0^t r_1(t-s)f(s)\,ds, \tag{1'}$$

$$u(t) = u_0 r_2(t) + \int_0^t r_2(t-s)f(s)\,ds, \tag{2'}$$

respectively. Here $r_1(t)$ and $r_2(t)$ are the resolvent kernels defined by

$$\tilde{r}_1(z) = \frac{\tilde{a}(z)}{1+\tilde{a}(z)}, \quad \tilde{r}_2(z) = \frac{1}{z+\tilde{a}(z)} \qquad (\operatorname{Re} z > x_0)$$

where $\tilde{a}(z)$ is the Laplace transform

$$\tilde{a}(z) \equiv \int_0^\infty e^{-zt}a(t)\,dt.$$

(This procedure may be justified when $f(t)$ is locally integrable and $\int_0^T |a(t)|\,dt \leqslant Ce^{\sigma T}$ $(0 < T < \infty)$ for some positive constants C, σ.)

When studying the asymptotic behavior as $t \to \infty$ of solutions of (1) or (2),

Received by the editors December 18, 1974 and, in revised form, March 5, 1975.
AMS (MOS) subject classifications (1970). Primary 45M05; Secondary 45M10, 42A68.
Key words and phrases. Volterra equations, resolvent, Fourier transform, Laplace transform.

as well as the behavior of solutions of related nonlinear equations, it is useful to find conditions on the kernel $a(t)$ which guarantee that the appropriate resolvent kernel $r_j(t)(j = 1,2)$ lies in $L^1(0,\infty)$.

When $a(t) \in L^1(0,\infty)$, a classical result due to Paley and Wiener [10, paragraph 18] states that $r_1(t) \in L^1(0,\infty)$ if and only if

$$\tilde{a}(z) \neq -1 \qquad (\operatorname{Re} z \geqslant 0). \tag{3}$$

Similarly, when $a(t)$ is integrable, a more recent result of Grossman and Miller [4] (also Shea [11]) yields $r_2(t) \in L^1(0,\infty)$ if and only if

$$\tilde{a}(z) \neq -z \quad (\operatorname{Re} z \geqslant 0). \tag{4}$$

However, many kernels $a(t)$ of importance in applications are not in $L^1(0,\infty)$. For example, Volterra integral equations with the kernel $a(t) = t^{-\frac{1}{2}}$ arise in the theory of superfluidity [6] as well as in problems of heat transfer between gases and solids [7].

Recently, Shea and Wainger [12] have used sophisticated methods from the theory of Laplace and Fourier transforms to obtain variants of the classical Wiener-Lévy theorem [10, p. 63]. It follows from their results that, for a large class of nonintegrable kernels $a(t)$, the resolvent $r_j(t)(j = 1,2)$ is in $L^1(0,\infty)$ whenever (3) or (4) holds. Other results of interest here concerning the behavior of $r_j(t)$ $(j = 1,2)$ for nonintegrable $a(t)$ have been obtained by Friedman [1], Miller [8] and Hannsgen [5]; see the Introduction of [12] for a discussion of these results.

THEOREM A (SHEA AND WAINGER). *Let* $a(t) = b(t) + \beta(t)$ *where*

(5) *$b(t)$ is nonnegative, nonincreasing and convex on $(0,\infty)$, with $b(t) \in L^1(0,1)$,*

and

$$\beta(t),\ t\beta(t) \in L^1(0,\infty). \tag{6}$$

Assume that $\varphi(w,z)$ is analytic on $S = \{(\tilde{a}(z),z)\colon \operatorname{Re} z \geqslant 0\}$ and at $(0,\infty)$ and $(\infty,0)$, and that $\varphi(0,\infty) = 0$. Then there exists $r(t) \in L^1(0,\infty)$ such that

$$\tilde{r}(z) = \varphi(\tilde{a}(z),z) \qquad (\operatorname{Re} z \geqslant 0). \tag{7}$$

(When $a(t) \notin L^1(0,\infty)$, $\tilde{a}(iy)$ is defined by $\tilde{a}(iy) = \lim_{x\to 0+} \tilde{a}(x+iy)$ for $-\infty < y < \infty$.)

Thus, the fact that (3) implies $r_1(t) \in L^1(0,\infty)$ for kernels having the form $a(t) = b(t) + \beta(t)$ with $b(t)$ and $\beta(t)$ satisfying hypotheses (5) and (6), respectively, follows from Theorem A with $\varphi_1(w,z) = w(1+w)^{-1}$. Similarly, Theorem A with $\varphi_2(w,z) = (z+w)^{-1}$ shows that, for kernels $a(t)$ of this form, $r_2(t) \in L^1(0,\infty)$ whenever (4) is satisfied.

In the same paper Shea and Wainger prove an alternate version of Theorem A in which hypotheses (5) and (6) are replaced by

$$a(t) = b + \beta(t) \quad \text{where } b \text{ is any constant}, \qquad \beta(t) \in L^1(0,\infty). \tag{8}$$

Our purpose here is to prove a sharpened version of Theorem A in which the perturbation term $\beta(t)$ is not required to satisfy the moment condition $t\beta(t) \in L^1(0, \infty)$. We have

THEOREM 1. *Theorem* A *remains valid with hypothesis* (6) *weakened to* $\beta(t) \in L^1(0, \infty)$.

The proof of Theorem 1 depends on Theorem A with special choices of the function $\varphi(w, z)$ together with techniques from transform theory.

Proof of Theorem 1. We may assume $b(t) \notin L^1(0, \infty)$ since otherwise Theorem 1 reduces to Shea and Wainger's alternate form of Theorem A with hypothesis (8).

Write

$$\text{(1.1)} \qquad \begin{aligned} \hat{a}(z) &= \tilde{a}(-iz) = \int_0^\infty e^{izt} a(t)\,dt \qquad (\operatorname{Im} z > 0), \\ \hat{a}(x) &= \lim_{y\to 0+} \hat{a}(x + iy) \qquad (-\infty < x < \infty). \end{aligned}$$

Also, when $r(t) \in L^1(-\infty, \infty)$, define $\hat{r}(x)$, by

$$\hat{r}(x) = \int_{-\infty}^{\infty} e^{ixt} r(t)\,dt \qquad (-\infty < x < \infty);$$

this notation is consistent with (1.1) when $r(t)$ vanishes on $(-\infty, 0)$.

As Shea and Wainger observe [12, §1], it suffices to find $r(t) \in L^1(-\infty, \infty)$ such that

$$\text{(1.2)} \qquad \hat{r}(x) = \varphi(\hat{a}(x), -ix) \qquad (-\infty < x < \infty).$$

Once $r(t)$ is found, a classical argument [10, p. 63] shows that $r(t) \equiv 0$ on $(-\infty, 0)$ and that (7) holds.

In order to find this $r(t)$ we write

$$\text{(1.3)} \qquad \varphi(\hat{a}(x), -ix) = \hat{\psi}_\delta(x)\varphi(\hat{a}(x), -ix) + [1 - \hat{\psi}_\delta(x)]\varphi(\hat{a}(x), -ix)$$

for $-\infty < x < \infty$. In equation (1.3) δ is a (small) positive number which will be selected in Step 1 below, and, for each positive number δ, $\psi_\delta(t) \in L^1(-\infty, \infty)$ is the function satisfying

$$\text{(1.4)} \qquad \hat{\psi}_\delta(x) = 1 \quad (|x| \leqslant \delta), \quad \hat{\psi}_\delta(x) = 0 \quad (|x| \geqslant 2\delta)$$

with $\hat{\psi}_\delta(x)$ linear otherwise. It is well known (see [2, p. 23]) that

$$\psi_\delta(t) = 2K_{2\delta}(t) - K_\delta(t)$$

where $K_\delta(t)$ is the usual Fejér kernel

$$K_\delta(t) = \pi^{-1}(1 - \cos \delta t)/\delta t^2 \qquad (-\infty < t < \infty).$$

The remainder of the proof consists of two steps.

Step 1. *There exists* $r_0(t) \in L^1(-\infty, \infty)$ *such that*

$$\hat{r}_0(x) = \hat{\psi}_\delta(x)\varphi(\hat{a}(x), -ix) \qquad (-\infty < x < \infty). \tag{1.5}$$

PROOF. Since $\varphi(w, z)$ is analytic at $(\infty, 0)$, there exists $\eta > 0$ such that

$$\varphi(w, z) = \sum_{m,n \geqslant 0} \beta_{mn} w^{-m} z^n \qquad (|w| \geqslant \eta^{-1}, |z| \leqslant \eta). \tag{1.6}$$

Since $|\hat{a}(x)| \to \infty$ as $|x| \to 0$ [5, Lemma 3], there exists $\delta_1 > 0$ so small that

$$|\hat{a}(x)| \geqslant \eta^{-1} \qquad (|x| \leqslant 2\delta_1). \tag{1.7}$$

Define $\varphi_3(w, z) = w[1 + z + w]^{-1}$. Since $b(t)$ satisfies (5), we have [5, p. 546] $\operatorname{Re} \tilde{b}(z) \geqslant 0$ for $\operatorname{Re} z \geqslant 0$, $z \neq 0$. Hence, Theorem A can be applied with $a(t) = b(t)$ and $\varphi = \varphi_3$ to obtain $r_3(t) \in L^1(0, \infty)$ such that

$$\tilde{r}_3(z) = \tilde{b}(z)[1 + z + \tilde{b}(z)]^{-1} \qquad (\operatorname{Re} z \geqslant 0).$$

Similarly, if we choose $\varphi_4(w, z) = [1 + z + w]^{-1}$ in Theorem A, we obtain $r_4(t) \in L^1(0, \infty)$ such that

$$\tilde{r}_4(z) = [1 + z + \tilde{b}(z)]^{-1} \qquad (\operatorname{Re} z \geqslant 0).$$

Therefore, if we extend the domains of the functions $r_3(t)$ and $r_4(t)$ to $(-\infty, \infty)$ by defining each function to be $\equiv 0$ on $(-\infty, 0)$, we may write

$$[\hat{a}(x)]^{-1} = \hat{r}_4(x)/(\hat{r}_3(x) + \hat{r}_4(x)\hat{\beta}(x)) \qquad (|x| \leqslant 2\delta_1).$$

Define the $L^1(-\infty, \infty)$ function $g(t)$ by

$$g(t) = r_3(t) + r_4 * \beta(t) \qquad (-\infty < t < \infty)$$

where the domain of $\beta(t)$ has been extended to $(-\infty, \infty)$ by defining $\beta(t) \equiv 0$ on $(-\infty, 0)$, and where $r_4 * \beta$ denotes the convolution

$$r_4 * \beta(t) = \int_{-\infty}^{\infty} r_4(t - s)\beta(s)\, ds \qquad (-\infty < t < \infty).$$

Observe that (1.7) together with $\hat{r}_3(0) = 1$, $\hat{r}_4(0) = 0$ and $|\hat{\beta}(0)| < \infty$ yield $\hat{g}(x) \neq 0$ for $|x| \leqslant 2\delta_1$. Hence, the Wiener-Lévy theorem for compact sets [2, p. 29] guarantees the existence of $g_1(t) \in L^1(-\infty, \infty)$ such that

$$\hat{g}_1(x) = [\hat{g}(x)]^{-1} \qquad (|x| \leqslant 2\delta_1).$$

Since $\hat{g}_1(0)\hat{r}_4(0) = 0$, there exist [2, p. 24] $\xi(t) \in L^1(-\infty, \infty)$ and $0 < \delta \leqslant \delta_1$ such that

$$\hat{\xi}(x) = \hat{g}_1(x)\hat{r}_4(x) = [\hat{a}(x)]^{-1} \qquad (|x| \leqslant 2\delta), \tag{1.8}$$

$$\|\xi\|_1 \leqslant \eta. \tag{1.9}$$

As Shea and Wainger observe [12, §2], the function $\theta_{2\delta}(t) = \psi'_{2\delta}(t) \in L^1(-\infty, \infty)$ satisfies

$$\begin{aligned} &\hat{\theta}_{2\delta}(x) = -ix \qquad (|x| \leqslant 2\delta), \quad \hat{\theta}_{2\delta}(x) = 0 \qquad (|x| \geqslant 4\delta), \\ &\|\theta_{2\delta}\|_1 \to 0 \quad \text{as } \delta \to 0. \end{aligned} \tag{1.10}$$

Assume that the δ (>0) in equation (1.8) is so small that, in addition to (1.9), we have

$$\|\theta_{2\delta}\|_1 \leqslant \eta. \tag{1.11}$$

Then, by (1.6), (1.8) and (1.10),

$$\hat{\psi}_\delta(x)\varphi(\hat{a}(x), -ix) = \sum_{m,n\geqslant 0} \beta_{mn}\hat{\psi}_\delta(x)[\hat{\xi}(x)]^m[\hat{\theta}_{2\delta}(x)]^n \quad (-\infty < x < \infty). \tag{1.12}$$

Using (1.6), (1.9), (1.11), (1.12), and the completeness of $L^1(-\infty, \infty)$, we see that $r_0(t) \in L^1(-\infty, \infty)$ defined by

$$r_0(t) = \sum_{m,n\geqslant 0} \beta_{mn}\psi_\delta * \xi^{m*} * \theta_{2\delta}^{n*}(t) \qquad (-\infty < t < \infty)$$

($\xi^{m*} = \xi * \cdots * \xi$ denotes the m-fold convolution in $L^1(-\infty, \infty)$) satisfies (1.5). This completes the proof of Step 1.

Step 2. *For* $\delta > 0$ *fixed as in Step* 1, *there exists* $r_c(t) \in L^1(-\infty, \infty)$ *such that*

$$\hat{r}_c(x) = [1 - \hat{\psi}_\delta(x)]\varphi(\hat{a}(x), -ix) \quad (-\infty < x < \infty). \tag{1.13}$$

PROOF. An examination of Shea and Wainger's proof of Theorem A [12, §1], shows that there exists $h(t) \in L^1(-\infty, \infty)$ such that

$$\hat{h}(x) = [1 - \hat{\psi}_{\delta/2}(x)]\hat{a}(x) \quad (-\infty < x < \infty).$$

(The existence of this $h(t)$ does not require that $t\beta(t) \in L^1(0, \infty)$.) Thus, using (1.4), we have

$$\varphi(\hat{a}(x), -ix) = \varphi(\hat{h}(x), -ix) \quad (|x| \geqslant \delta). \tag{1.14}$$

But since $h(t) \in L^1(-\infty, \infty)$, Shea and Wainger's proof of the alternate version of Theorem A with hypothesis (8) [12, §2] establishes the existence of $r_c(t) \in L^1(-\infty, \infty)$ such that

$$\hat{r}_c(x) = [1 - \hat{\psi}_\delta(x)]\varphi(\hat{h}(x), -ix) \quad (-\infty < x < \infty).$$

Clearly (1.13) now follows from (1.4) and (1.14), and the proof of Step 2 is complete.

Finally, $r(t) = r_0(t) + r_c(t)$ is the $L^1(-\infty, \infty)$ function which satisfies (1.2); hence, by the remark at the beginning of this section, the proof of Theorem 1 is complete.

ACKNOWLEDGEMENT. The authors wish to thank Professor Richard K. Miller of Iowa State University for helpful discussions concerning the subject of this paper.

REFERENCES

1. A. Friedman, *On integral equations of Volterra type*, J. Analyse Math. **11**(1963), 381–413. MR **28** #1458.

2. R. R. Goldberg, *Fourier transforms*, Cambridge Tracts in Math. and Math. Phys., no. 52, Cambridge Univ. Press, New York, 1961. MR **22** #11254.

3. S. I. Grossman and R. K. Miller, *Perturbation theory for Volterra integrodifferential systems*, J. Differential Equations **8**(1970), 457–474. MR **42** #4988.

4. ———, *Nonlinear Volterra integrodifferential systems with L^1-kernels*, J. Differential Equations **13** (1973), 551–566.

5. K. B. Hannsgen, *Indirect abelian theorems and a linear Volterra equation*, Trans. Amer. Math. Soc. **142** (1969), 539–555. MR **39** #7364.

6. N. Levinson, *A nonlinear Volterra equation arising in the theory of superfluidity*, J. Math. Anal. Appl. **1** (1960), 1–11.

7. W. R. Mann and F. Wolf, *Heat transfer between solids and gases under nonlinear boundary conditions*, Quart. Appl. Math. **9** (1951), 163–184. MR **13**, 134.

8. R. K. Miller, *On Volterra integral equations with nonnegative integrable resolvents*, J. Math. Anal. Appl. **22** (1968), 319–340. MR **37** #3291.

9. ———, *Nonlinear Volterra integral equations*, Lecture Note Series, Benjamin, New York, 1971.

10. R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, Amer. Math. Soc. Colloq. Publ., vol. 19, Amer. Math. Soc., Providence, R. I., 1934.

11. D. F. Shea, Unpublished manuscript.

12. D. F. Shea and S. Wainger, *Variants of the Wiener-Lévy theorem, with applications to stability problems for some Volterra integral equations*, Amer. J. Math. **97** (1975), 312–343.

Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37916 (Current address of G. S. Jordan)

Department of Mathematics, Iowa State University, Ames, Iowa 50010

Department of Mathematics, University of Missouri, Columbia, Missouri 65201 (Current address of R. L. Wheeler)