

## EXTREME OPERATORS IN THE UNIT BALL OF $L(C(X), C(Y))$ OVER THE COMPLEX FIELD<sup>1</sup>

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**ABSTRACT.** Assume that  $X$  and  $Y$  are compact Hausdorff spaces and that  $C(X)$  and  $C(Y)$  are the Banach spaces of continuous complex-valued functions on  $X$  and  $Y$ , respectively.  $L(C(X), C(Y))$  is the space of bounded linear operators from  $C(X)$  to  $C(Y)$ . If  $E$  is a Banach space, then  $S(E)$  is the closed unit ball in  $E$ . An operator  $T$  in  $S(L(C(X), C(Y)))$  is *nice* if  $T^*(\text{ext } S(C(Y)^*)) \subset \text{ext } S(C(X)^*)$ . For each  $y \in Y$ ,  $\epsilon_y$  denotes point mass at  $y$ . The main theorem states that if  $T$  is extreme in  $S(L(C(X), C(Y)))$  and  $\|T^*(\epsilon_y)\| = 1$  for all  $y \in Y$ , then  $T$  is nice. Other theorems are proved by using the same techniques as in the proof of the main theorem.

Denote the unit ball of the Banach space  $E$  by  $S(E)$  and the set of extreme points of  $S(E)$  by  $\text{ext } S(E)$ .

Suppose  $E$  and  $F$  are Banach spaces. We will denote the Banach space of bounded linear operators from  $E$  into  $F$  by  $L(E, F)$ . The dual of the Banach space  $E$  will be denoted by  $E^*$ . If  $T \in L(E, F)$ , then  $T^*$  will be its adjoint map in  $L(F^*, E^*)$ .

The term *real (complex) scalars* in connection with a theorem or definition will mean that the vector spaces being discussed are over the real (complex) number field, and if any of the vector spaces are function spaces, then the functions will be real (complex) valued. Throughout this paper, we shall assume that the scalars are complex unless we state otherwise.

The symbols  $X$  and  $Y$  will always denote compact Hausdorff spaces, and  $C(X)$  will denote the Banach space of continuous functions on  $X$ , normed by the usual supremum norm. If  $u \in C(X)^*$ , we may and shall treat  $u$  both as a measure and as a linear functional. Furthermore,

$$\text{ext } S(C(X)^*) = \{\lambda \epsilon_x : |\lambda| = 1, x \in X\}$$

where  $\epsilon_x$  denotes point mass at  $x \in X$ . (See Arens and Kelley [2].) If the scalars are real and  $u \in C(X)^*$ , then  $u$  can be written as  $u = u^+ - u^-$  where  $u^+$  and  $u^-$  are the positive and negative parts of  $u$ .

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**DEFINITION.** An operator  $T \in S(L(E, F))$  is a *nice operator* if  $T^*(\text{ext } S(F^*)) \subset \text{ext } (S(E^*))$ .

If an operator is nice in  $S(L(E, F))$ , then it is obviously extreme.

In this paper we shall study extreme operators in  $S(L(C(X), C(Y)))$ . We shall abbreviate  $L(C(X), C(Y))$  by  $L(X, Y)$  and  $S(L(C(X), C(Y)))$  by  $S(X, Y)$ . Also, if  $T \in L(E, C(Y))$  and  $y \in Y$ , then we shall abbreviate  $T^*(\epsilon_y)$  by  $T^*(y)$ .

**CONJECTURE.** If  $T$  is extreme in  $S(X, Y)$ , then  $T$  is nice.

This conjecture has been proven true under various additional assumptions. We list some of the published results. The conjecture is true if:

(1) The scalars are real and  $X$  is metric (Blumenthal, Lindenstrauss and Phelps [3]).

(2) The scalars are real and  $\|T^*(y)\| = 1$  for all  $y \in Y$  [3].

(3) The scalars are real and  $T$  is compact [3].

(4) The scalars are real,  $X$  is Eberlein-compact, and  $Y$  is metric (Amir and Lindenstrauss [1]).

(5) The scalars are real,  $Y$  is separable, and  $T$  is weakly compact (Corson and Lindenstrauss [4]).

(6)  $Y$  is extremally disconnected (Sharir [8]).

Our main result is the complex version of (2) above. The same methods lead to a new and shorter proof of (6) and a similar proof of a closely related result (Theorem 1.4).

The following proposition gives a characterization of the space  $L(E, C(Y))$  which will be used throughout the paper. The proof can be found in [5, p. 490].

**1.1. PROPOSITION.** *Suppose that  $E$  is a Banach space and that  $Y$  is a compact Hausdorff space. Then there is an isometry between  $L(E, C(Y))$  and the space of weak\* continuous functions from  $Y$  into  $E^*$ . The isometry is defined by assigning to each  $T \in L(E, C(Y))$  the map  $y \rightarrow T^*(y)$ , where  $\|T\| = \sup\{\|T^*(y)\|: y \in Y\}$ . Furthermore, the subspace of all weakly compact operators in  $L(E, C(Y))$  is isometric to the space of all weakly continuous functions from  $Y$  into  $E^*$ , and the subspace of all compact operators in  $L(E, C(Y))$  is isometric to the space of all norm continuous functions from  $Y$  into  $E^*$ .*

The proof of the next lemma is straightforward. (See, for example, Sharir [8].)

**1.2. LEMMA.** *Suppose that  $T$  is extreme in  $S(X, Y)$ . Let  $H = \{y \in Y: \|T^*(y)\| = 1\}$ . Then  $H$  is a dense  $G_\delta$  subset of  $Y$ .*

**DEFINITIONS.** A compact Hausdorff space  $Y$  is *extremally disconnected* if the closure of every open set is open. It is *basically disconnected* if the closure of every open  $F_\sigma$ -set is open.

**1.3. THEOREM.** *Let  $T$  be extreme in  $S(X, Y)$  and suppose that*

- (i)  $\|T^*(y)\| = 1$  for all  $y$  in  $Y$ , or
- (ii)  $Y$  is extremally disconnected.

*Then  $T$  is nice.*

**PROOF.** Assume that  $T$  is not nice. Then there exist a point  $y_0 \in Y$  and

distinct points  $x_1$  and  $x_2$  in  $X$  such that  $x_1$  and  $x_2$  are in the support of  $|T^*(y_0)|$ .

There exists a function  $g \in C(X)$  such that  $0 \leq g \leq 1$  and  $g(x_1) = 1$  and  $g(x_2) = 0$ . Clearly,  $g$  is not constant a.e. with respect to  $|T^*(y_0)|$ .

Define operators  $U_1$  and  $U_2$  in  $S(X, Y)$  by

$$U_1(f) = T(fg) \quad \text{for all } f \in C(X),$$

$$U_2(f) = T(f(1 - g)) \quad \text{for all } f \in C(X).$$

For each  $y \in Y$  consider the polar decomposition of  $T^*(y)$ . (See Rudin [7].) That is,  $dT^*(y) = h_y d|T^*(y)|$  where  $|h_y(x)| = 1$  for all  $x \in X$ . Now, for each  $y \in Y$ ,

$$dU_1^*(y) = gh_y d|T^*(y)|, \quad dU_2^*(y) = (1 - g)h_y d|T^*(y)|.$$

Define functions  $\alpha_1$  and  $\alpha_2$  from  $H = \{y \in Y: \|T^*(y)\| = 1\}$  into  $[0, 1]$  by  $\alpha_1(y) = \|U_1^*(y)\|$  and  $\alpha_2(y) = \|U_2^*(y)\|$  for all  $y \in H$ . Note that

$$\begin{aligned} \alpha_1(y) + \alpha_2(y) &= \int gd|T^*(y)| + \int (1 - g)d|T^*(y)| \\ &= \int d|T^*(y)| = 1 \end{aligned}$$

for all  $y \in H$ . It follows readily from this fact, together with the weak\* lower semicontinuity of the norm in  $C(X)^*$ , that  $\alpha_1$  and  $\alpha_2$  are continuous on  $H$ .

By Lemma 1.2,  $H$  is dense in  $Y$ . Thus, define the continuous functions  $\bar{\alpha}_1, \bar{\alpha}_2$  from  $Y$  into  $[0, 1]$  by  $\bar{\alpha}_k = \alpha_k$  ( $k = 1, 2$ ) in case (i) or as extensions of  $\alpha_1, \alpha_2$  in case (ii). (See [6, p. 96].) Note that  $\bar{\alpha}_1(y) + \bar{\alpha}_2(y) = 1$  for all  $y \in Y$ .

Consider the operator  $U$  in  $S(X, Y)$  defined by

$$U^*(y) = \bar{\alpha}_2(y)U_1^*(y) - \bar{\alpha}_1(y)U_2^*(y)$$

for all  $y \in Y$ . Note that if  $y \in H$ , then

$$\begin{aligned} \|T^*(y) \pm U^*(y)\| &= \|(1 \pm \|U_2^*(y)\|)U_1^*(y) + (1 \mp \|U_1^*(y)\|)U_2^*(y)\| \\ &\leq (1 \pm \|U_2^*(y)\|)\|U_1^*(y)\| + (1 \mp \|U_1^*(y)\|)\|U_2^*(y)\| \\ &= \|U_1^*(y)\| + \|U_2^*(y)\| = 1. \end{aligned}$$

Hence, by the lower semicontinuity of the norm under the weak\* topology,  $\|T^*(y) \pm U^*(y)\| \leq 1$  for all  $y \in Y$ . Since  $T$  is extreme in  $S(X, Y)$ , then  $U^*(y) = 0$  for all  $y \in Y$ . Hence,

$$0 = \|U^*(y)\| = \int |\bar{\alpha}_2(y)g - \bar{\alpha}_1(y)(1 - g)|d|T^*(y)|$$

for all  $y \in Y$ . This implies that  $\bar{\alpha}_2(y)g - \bar{\alpha}_1(y)(1 - g) = 0$  a.e. ( $|T^*(y)|$ ) for all  $y \in Y$ . However,  $\bar{\alpha}_2(y)g - \bar{\alpha}_1(y)(1 - g) = g - \bar{\alpha}_1(y)$ . Thus,  $g = \bar{\alpha}_1(y)$  a.e. ( $|T^*(y)|$ ) for all  $y \in Y$ . This is a contradiction as  $g$  is not constant a.e. ( $|T^*(y_0)|$ ), and the proof is complete.

We now prove a similar result for  $X$  metric and  $Y$  basically disconnected.

1.4. THEOREM. *Suppose that  $X$  is metric and  $Y$  is basically disconnected. If  $T$  is extreme in  $S(X, Y)$ , then  $T$  is nice.*

PROOF. If  $T$  is not nice, define  $y_0, x_1, x_2, g, U_1$ , and  $U_2$  as in the proof of Theorem 1.3. Since  $X$  is metric,  $C(X)$  is separable. Thus, there exists a countable dense subset  $D \subset S(C(X))$  such that

$$\|U_1^*(y)\| = \sup_{f \in D} |U_1(f)(y)|,$$

$$\|U_2^*(y)\| = \sup_{f \in D} |U_2(f)(y)|$$

for all  $y \in Y$ . Furthermore, for each  $f \in D$  and  $y \in Y$ ,

$$|U_1(f)(y)| \leq 1 \quad \text{and} \quad |U_2(f)(y)| \leq 1.$$

Since  $Y$  is basically disconnected,  $C(Y)$  is  $\sigma$ -order complete [6]. Thus,  $\sup_{f \in D} |U_1(f)| = \bar{\alpha}_1$  and  $\sup_{f \in D} |U_2(f)| = \bar{\alpha}_2$  exist in  $C(Y)$ . Also,  $\bar{\alpha}_1(y) = \|U_1^*(y)\|$  and  $\bar{\alpha}_2(y) = \|U_2^*(y)\|$  for all  $y \in H$  because the point supremum and the lattice supremum are equal at points of continuity of the point supremum. The proof continues as in the proof of Theorem 1.3.

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