KERNELS IN LATTICE-ORDERED GROUPS

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Abstract. It is shown that a lattice-ordered group has a largest representable convex \( l \)-subgroup and a largest normal valued convex \( l \)-subgroup. Other kernels are discussed.

1. Introduction. Throughout this note \( G \) will denote a lattice-ordered group \((l \)-group\). In [7, Theorem 1.6] J. Martinez proved that if \( G \) is a representable lattice-ordered group, then \( G \) had a largest hyperarchimedean (also called epiarchimedean) convex \( l \)-subgroup (the hyperarchimedean kernel). P. Conrad [5, Theorem 3.5] extended this result to an arbitrary \( l \)-group. In [6] G. O. Kenny announced that a representable \( l \)-group had a largest archimedean convex \( l \)-subgroup (the archimedean kernel) and R. Redfield [8, Corollary 2.2] extended this result, again by dropping the hypothesis of representability. In this note we prove that if \( G \) is an \( l \)-group, then \( G \) has a largest representable convex \( l \)-subgroup (the representable kernel) and a largest normal valued convex \( l \)-subgroup (the normal valued kernel).

If \( X \) is a subset of \( G \), then \([X] \) denotes the subgroup of \( G \) generated by \( X \), and \( \mathcal{C}(G) \) denotes the lattice of convex \( l \)-subgroups of \( G \). The reader is referred to [4] for the standard terminology and results in \( l \)-groups.

2. Kernels. An \( l \)-group is said to be normal valued if each regular subgroup is normal in the convex \( l \)-subgroup that covers it. If \( G \) is representable, then \( G \) is normal valued [1, Corollary 3.2]. Let \( \mathcal{K}(G) = \{H \mid H \) is a hyperarchimedean convex \( l \)-subgroup of \( G \} \), \( \mathcal{A}(G) = \{H \mid H \) is an archimedean convex \( l \)-subgroup of \( G \} \), \( \mathcal{R}(G) = \{H \mid H \) is a representable convex \( l \)-subgroup of \( G \} \), and \( \mathcal{N}(G) = \{H \mid H \) is a normal valued convex \( l \)-subgroup of \( G \} \). Then \( \mathcal{K}(G) \subseteq \mathcal{A}(G) \subseteq \mathcal{R}(G) \subseteq \mathcal{N}(G) \) and, as mentioned in the introduction, \( \mathcal{K}(G) \) and \( \mathcal{A}(G) \) are principal ideals of \( \mathcal{C}(G) \).

Theorem 2.1. Let \( G \) be an \( l \)-group and \( C \in \mathcal{C}(G) \).

(i) If \( H, J \in \mathcal{A}(G) \) and \( C \subseteq H \), then \( C, H + J \in \mathcal{A}(G) \).

(ii) If \( H, J \in \mathcal{R}(G) \) and \( C \subseteq H \), then \( H + J = [H \cup J] \) and \( C, H + J \in \mathcal{R}(G) \).

(iii) If \( H, J \in \mathcal{N}(G) \) and \( C \subseteq H \), then \( H + J = [H \cup J] \) and \( C, H + J \in \mathcal{N}(G) \).

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Proof. (i) To show that $H + J$ is abelian, it suffices to show that if $h \in H^+$ and $j \in J^+$, then $h + j = j + h$. Now $h \land j \in H \cap J$ and, hence, commutes with $h$ and $j$. Since $h \lor j = h - (h \land j) + j = j \lor h$, it follows that $h + j = j + h$.

(ii) To prove that $[H \cup J]$ is representable, it suffices to show that each minimal prime subgroup of $[H \cup J]$ is normal in $[H \cup J]$ [1, Theorem 3.1]. If $M$ is a minimal prime subgroup of $[H \cup J]$, then $H \not\subseteq M$ or $J \not\subseteq M$.

Case 1. $H \not\subseteq M$ and $J \subseteq M$. Then $J \subseteq n(M)$, the normalizer of $M$ in $[H \cup J]$. By [2, Theorem 3.5], $H \cap M$ is a minimal prime subgroup of $H$ and, since $H$ is representable, $H \cap M$ is normal in $H$ [1, Theorem 3.1]. Hence, by [2, Theorem 3.6], $H \subseteq n(M)$ and so $[H \cup J] = n(M)$.

The cases $H \subseteq M$ and $J \not\subseteq M$, and $H \not\subseteq M$ and $J \subseteq M$ are similar to Case 1 and are omitted. Trivially, if $C \subseteq H$, then $C \subseteq R(G)$. It will be shown in (iii) that $H + J = [H \cup J]$, since $R(G) \subseteq R(G)$.

(iii) Let $T$ be a regular subgroup of $[H \cup J]$, $T^*$ be the convex $l$-subgroup of $[H \cup J]$ that covers $T$, and let $0 \leq t \in T^*$ with $t \not\in T$. Then $t = h_1 + j_1 + \cdots + h_n + j_n$ where $h_i \in H^+$ and $j_i \in J^+$ (see [4, p. 1.7]). Thus, for some $1 \leq k \leq n$, $h_k \not\in T$ or $j_k \not\in T$. Without loss of generality, we may assume that $h_k \not\in T$. Then $H \cap T$ is a regular subgroup of $H$, $h_k \not\in H \cap T$, $H \cap T^*$ is the convex $l$-subgroup of $H$ that covers $H \cap T$, and $h_k \not\in H \cap T^*$ [2, Theorem 3.5]. Since $H \subseteq R(G)$, $H \cap T$ is normal in $H \cap T^*$. Hence, by [2, Theorem 3.6], $T$ is normal in $T^*$. Therefore, $[H \cup J] \subseteq R(G)$. By [9, Theorem 3], $H + J = [H \cup J]$.

If $C \subseteq H$, then $C \subseteq R(G)$ [9, Corollary 1, p. 342].

Corollary 2.2. If $G$ is an $l$-group and $\mathcal{K} \in \{R(G), R(G), R(G)\}$, then $\bigcup \mathcal{K} \in \mathcal{K}$ and so $\mathcal{K}$ is a principal ideal of $R(G)$. Moreover, $\bigcup \mathcal{K}$ is invariant under all $l$-automorphisms of $G$.

Let $I(G)$, $D(G)$, and $R(G)$ denote the ideal radical, the distributive radical, and the radical of $C$ respectively (see [3] or [4, Chapter 5]), $\mathcal{I}(G) = \{H \in \mathcal{C}(G) \cap I(H) = \{0\}, \mathcal{D}(G) = \{H \in \mathcal{C}(G) \cap D(H) = \{0\}\}$, and $\mathcal{R}(G) = \{H \in \mathcal{C}(G) \cap R(H) = \{0\}\}$. If $H \subseteq \mathcal{C}(G)$ and $\{H \subseteq \mathcal{C}(G) \cap R(H) = \{0\}\}$. The ideal radical and the radical have the same properties.

Theorem 2.3. If $G$ is an $l$-group and $\mathcal{K} \in \{R(G), D(G), \mathcal{I}(G)\}$, then $\bigcup \mathcal{K} \subseteq \mathcal{K}$, $\bigcup \mathcal{K}$ is invariant under all $l$-automorphisms of $G$, and $\mathcal{K}$ is a principal ideal of $\mathcal{C}(G)$.

References


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