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KERNELS IN LATTICE-ORDERED GROUPS

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ABSTRACT. It is shown that a lattice-ordered group has a largest representable convex l-subgroup and a largest normal valued convex l-subgroup. Other kernels are discussed.

1. **Introduction.** Throughout this note G will denote a lattice-ordered group (l-group). In [7, Theorem 1.6] J. Martinez proved that if G is a representable lattice-ordered group, then G had a largest hyperarchimedean (also called epiarchimedean) convex l-subgroup (the hyperarchimedean kernel). P. Conrad [5, Theorem 3.5] extended this result to an arbitrary l-group. In [6] G. O. Kenny announced that a representable l-group had a largest archimedean convex l-subgroup (the archimedean kernel) and R. Redfield [8, Corollary 2.2] extended this result, again by dropping the hypothesis of representability. In this note we prove that if G is an l-group, then G has a largest representable convex l-subgroup (the representable kernel) and a largest normal valued convex l-subgroup (the normal valued kernel).

If X is a subset of G, then $[X]$ denotes the subgroup of G generated by X, and $\mathcal{C}(G)$ denotes the lattice of convex l-subgroups of G. The reader is referred to [4] for the standard terminology and results in l-groups.

2. **Kernels.** An l-group is said to be normal valued if each regular subgroup is normal in the convex l-subgroup that covers it. If G is representable, then G is normal valued [1, Corollary 3.2]. Let $\mathcal{H}(G) = \{H \mid H$ is a hyperarchimedean convex l-subgroup of $G\}$, $\mathrm{Ar}(G) = \{H \mid H$ is an archimedean convex l-subgroup of $G\}$, $\mathcal{A}(G) = \{H \mid H$ is an abelian convex l-subgroup of $G\}$, $\mathcal{R}(G) = \{H \mid H$ is a representable convex l-subgroup of $G\}$, and $\mathcal{N}(G) = \{H \mid H$ is a normal valued convex l-subgroup of $G\}$. Then $\mathcal{H}(G) \subseteq \mathrm{Ar}(G) \subseteq \mathcal{A}(G) \subseteq \mathcal{R}(G) \subseteq \mathcal{N}(G)$ and, as mentioned in the introduction, $\mathcal{H}(G)$ and $\mathrm{Ar}(G)$ are principal ideals of $\mathcal{C}(G)$.

THEOREM 2.1. *Let G be an l-group and $C \in \mathcal{C}(G)$.*

(i) *If $H, J \in \mathcal{A}(G)$ and $C \subseteq H$, then $C, H + J \in \mathcal{A}(G)$.*

(ii) *If $H, J \in \mathcal{R}(G)$ and $C \subseteq H$, then $H + J = [H \cup J]$ and $C, H + J \in \mathcal{R}(G)$.*

(iii) *If $H, J \in \mathcal{N}(G)$ and $C \subseteq H$, then $H + J = [H \cup J]$ and $C, H + J \in \mathcal{N}(G)$.*

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PROOF. (i) To show that $H + J$ is abelian, it suffices to show that if $h \in H^+$ and $j \in J^+$, then $h + j = j + h$. Now $h \wedge j \in H \cap J$ and, hence, commutes with h and j. Since $h \vee j = h - (h \wedge j) + j = j \vee h$, it follows that $h + j = j + h$.

(ii) To prove that $[H \cup J]$ is representable, it suffices to show that each minimal prime subgroup of $[H \cup J]$ is normal in $[H \cup J]$ [1, Theorem 3.1]. If M is a minimal prime subgroup of $[H \cup J]$, then $H \not\subseteq M$ or $J \not\subseteq M$.

Case 1. $H \not\subseteq M$ and $J \subseteq M$. Then $J \subseteq n(M)$, the normalizer of M in $[H \cup J]$. By [2, Theorem 3.5], $H \cap M$ is a minimal prime subgroup of H and, since H is representable, $H \cap M$ is normal in H [1, Theorem 3.1]. Hence, by [2, Theorem 3.6], $H \subseteq n(M)$ and so $[H \cup J] = n(M)$.

The cases $H \subseteq M$ and $J \not\subseteq M$, and $H \not\subset M$ and $J \not\subseteq M$ are similar to Case 1 and are omitted. Trivially, if $C \subseteq H$, then $C \in \mathfrak{R}(G)$. It will be shown in (iii) that $H + J = [H \cup J]$, since $\mathfrak{R}(G) \subseteq \mathfrak{N}(G)$.

(iii) Let T be a regular subgroup of $[H \cup J]$, T^* be the convex l-subgroup of $[H \cup J]$ that covers T, and let $0 < t \in T^*$ with $t \notin T$. Then $t = h_1 + j_1 + \cdots + h_n + j_n$ where $h_i \in H^+$ and $j_i \in J^+$ (see [4, p. 1.7]). Thus, for some $1 \leqslant k \leqslant n$, $h_k \notin T$ or $j_k \notin T$. Without loss of generality, we may assume that $h_k \notin T$. Then $H \cap T$ is a regular subgroup of H, $h_k \notin H \cap T$, $H \cap T^*$ is the convex l-subgroup of H that covers $H \cap T$, and $h_k \in H \cap T^*$ [2, Theorem 3.5]. Since $H \in \mathfrak{N}(G)$, $H \cap T$ is normal in $H \cap T^*$. Hence, by [2, Theorem 3.6], T is normal in T^*. Therefore, $[H \cup J] \in \mathfrak{N}(G)$. By [9, Theorem 3], $H + J = [H \cup J]$.

If $C \subseteq H$, then $C \in \mathfrak{N}(G)$ [9, Corollary 1, p. 342].

COROLLARY 2.2. *If G is an l-group and $\mathfrak{X} \in \{\mathfrak{A}(G), \mathfrak{R}(G), \mathfrak{N}(G)\}$, then $\bigcup \mathfrak{X} \in \mathfrak{X}$ and so $\mathfrak{X}$ is a principal ideal of $\mathcal{C}(G)$. Moreover, $\bigcup \mathfrak{X}$ is invariant under all l-automorphisms of G.*

Let $I(G)$, $D(G)$, and $R(G)$ denote the ideal radical, the distributive radical, and the radical of C respectively (see [3] or [4, Chapter 5]), $\mathfrak{I}(G) = \{H \mid H \in \mathcal{C}(G)$ and $I(H) = \{0\}\}$, $\mathfrak{D}(G) = \{H \mid H \in \mathcal{C}(G)$ and $D(H) = \{0\}\}$, and $\mathrm{Ra}\,(G) = \{H \mid H \in \mathcal{C}(G)$ and $R(H) = \{0\}\}$. If $H \in \mathcal{C}(G)$ and $\{H_\lambda \mid \lambda \in \Lambda\} \subseteq \mathcal{C}(G)$, then $D(H) = H \cap D(G)$ and $D([\bigcup H_\lambda]) = [\bigcup D(H_\lambda)]$ [3, p. 126]. The ideal radical and the radical have the same properties.

THEOREM 2.3. *If G is an l-group and $\mathfrak{X} \in \{\mathrm{Ra}\,(G), \mathfrak{D}(G), \mathfrak{I}(G)\}$, then $\bigcup \mathfrak{X} \in \mathfrak{X}$, $\bigcup \mathfrak{X}$ is invariant under all l-automorphisms of G, and $\mathfrak{X}$ is a principal ideal of $\mathcal{C}(G)$.*

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