THE LARGEST PROPER VARIETY OF LATTICE ORDERED GROUPS

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Abstract. If a lattice ordered group $G$ satisfies any identical relation, other than those satisfied by every lattice ordered group, then $G$ is normal valued, and hence satisfies the relation $ab < b^2a^2$ for all $a, b \geq e$.

A lattice ordered group ($l$-group) $G$ is said to be normal valued if whenever $M$ is a convex $l$-subgroup maximal with respect to missing a fixed element $g \in G$, and $K$ the smallest convex $l$-subgroup of $G$ containing $M$ and $g$, then $M$ is a normal subgroup of $K$. It was shown by Wolfenstein [5] that the normal valued $l$-groups are characterized by the property that $ab < b^2a^2$ for all $a, b \geq e$, and thus constitute a variety, or equationally defined class (the inequality is equivalent to the equation $|x||y||x|^{-2}|y|^{-2} \vee e = e$, where $|z| = (z \vee z^{-1})$). It has been observed that the variety of normal valued $l$-groups is very large; all of the many varieties studied by Martinez [2] are contained in the normal valued variety (with the exception of the variety of all $l$-groups). It will be shown here that every property variety of lattice ordered groups is contained in the normal valued variety. This sheds a new light on several of Martinez's results, and shows that certain $l$-groups are generic. For example, if $A(R)$ denotes the $l$-group of all order preserving permutations of the real line, and if $A(R)$ satisfies an identical relation, then every $l$-group must satisfy that relation.

Theorem. If a lattice ordered group $G$ satisfies an identical relation which is not satisfied by every lattice ordered group, then $G$ is normal valued.

The Theorem will be proved in a sequence of lemmas.

If an $l$-group $H$ is an $l$-subgroup of the $l$-group $A(S)$ of all order preserving permutations of a totally ordered set $S$, $H$ is said to be $o$-$2$-transitive on $S$ if whenever $s_1 < s_2$, $t_1 < t_2$ are members of $S$, there exists $h \in H$ such that $s_1 h = t_1$. It follows easily that any such $H$ must in fact be $o$-$n$-transitive in the sense that whenever $s_1 < s_2 < \cdots < s_n$, $t_1 < t_2 < \cdots < t_n$ are members of $S$, there exists $h \in H$ such that $s_i h = t_i$ [4, Lemma 4]. Again, $H$ is said to be $o$-primitive on $S$ if $H$ acts transitively on $S$ and the stabilizer subgroups $H_s = \{h \in H | sh = s\}$ are maximal convex $l$-subgroups. Finally, $H$ is periodic on $S$ if $H$ is periodic on $S$ and there is a period $f \in A(S)$, where $S$ is the Dedekind completion of $S$, such that $fh = hf$ for all $h \in H$, where $H \subseteq A(S)$.
in the natural way, and \( f \) has coterminal orbits. It was shown by McCleary \[3\] that if \( H \) is \( o \)-primitive on \( S \) then either \( H_s = \{e\} \) for every \( s \in S \), or \( H \) is \( o \)-2-transitive on \( S \), or \( H \) is periodic on \( S \). In the latter case, \( H_t \) acts faithfully and \( o \)-2-transitively on the interval \((s, s_f)\) of \( S \), where \( f \) is the period, and \( s \) is any member of \( S \).

**Lemma 1.** If the \( l \)-subgroup \( H \) of \( A(S) \) is \( o \)-primitive on \( S \) but \( H_s \neq \{e\} \) for some \( s \in S \), then \( H \) contains an \( l \)-subgroup which is \( o \)-2-transitive on some totally ordered set.

Now suppose that \( G \) is an \( l \)-group which is not normal valued. There exists, then, a convex \( l \)-subgroup \( M \) of \( G \), maximal with respect to missing some element \( g \in G \), such that \( M \) is not a normal subgroup of its cover \( K \). The intersection \( \cap k^{-1}Mk \) of all the conjugates of \( M \) in \( K \) is an \( l \)-ideal of \( K \), and the \( l \)-group \( H = K/\cap k^{-1}Mk \) is \( l \)-isomorphic to an \( o \)-primitive \( l \)-subgroup of order preserving permutations of the totally ordered set \( S \) of right cosets of \( M \) in \( K \). In this representation, \( H_s = M \) for some \( s \in S \), and so \( H_s \neq \{e\} \) since \( M \) is not normal in \( K \). By Lemma 1, \( H \) contains an \( l \)-subgroup which is \( o \)-2-transitive on some set. This subgroup must belong to any variety that contains \( G \). Thus:

**Lemma 2.** If \( G \) is an \( l \)-group which is not normal valued, then every variety containing \( G \) contains an \( l \)-group which is an \( o \)-2-transitive \( l \)-group of order preserving permutations of some totally ordered set.

Let \( X \) be a countably infinite set of letters and \( X^{-1} = \{x^{-1} | x \in X\} \) a set disjoint from \( X \). Let \( F \) be the free \( l \)-group on \( X \). The elements of \( F \) may be written in the form \( \bigwedge A \land B \prod_{\Gamma} x_{a\beta y} \) where \( A, B, \Gamma \) are finite index sets, \( \Gamma = \{1, 2, \ldots, n\} \), \( x_{a\beta y} \in X \cup X^{-1} \cup \{e\} \), \( \prod \) indicates the group operation, and \( \lor \) and \( \land \) the lattice operations. There is, in general, nothing unique about the form of a given element of \( F \). An identical relation is, then, a formal expression \( w = e \), where \( w \in F \). An \( l \)-group \( H \) is said to satisfy the identical relation \( w = e \), where \( w \) has the form above, if for every substitution \( x_{a\beta y} \mapsto h_{a\beta y} \) by elements of \( H \), we have \( e = \bigwedge A \land B \prod_{\Gamma} h_{a\beta y} \), where it is understood that if \( h \) is substituted for one occurrence of \( x \), \( h^{-1} \) must also be substituted for all other occurrences of the same \( x \), \( h^{-1} \) for \( x^{-1} \), and \( e \) (in \( H \)) for \( e \) (in \( F \)), and \( \lor, \land, \prod \) indicate the lattice and group operations in \( H \).

**Lemma 3.** Let \( H \) be a nontrivial \( o \)-2-transitive \( l \)-group of order preserving permutations of a totally ordered set \( S \), and \( w \in F \) not the identity element of the free \( l \)-group \( F \). Then \( H \) does not satisfy the identical relation \( w = e \).

To prove Lemma 3, it may first be assumed that \( F \) is an \( l \)-group of order preserving permutations of a totally ordered set \( T \) \[1\]. There must exist a point \( t \in T \) such that \( tw \neq t \). Let \( w = \bigwedge A \land B \prod_{\Gamma} x_{a\beta y}, \Gamma = \{1, 2, \ldots, n\} \). For each \( (\alpha, \beta) \in A \times B \), define \( t(\alpha, \beta, 0) = t \), and for \( 1 \leq \gamma \leq n \), \( t(\alpha, \beta, \gamma) = t(\alpha, \beta, \gamma - 1) x_{\alpha\beta y} \). Now for each \( x \in X \) occurring in \( w \), and each \( (\alpha, \beta) \), let \( P_{\alpha\beta}(x) = (\gamma | x = x_{\alpha\beta y} \) and \( N_{\alpha\beta}(x) = (\gamma | x^{-1} = x_{\alpha\beta y} \). Then if \( \gamma \in P_{\alpha\beta}(x), t(\alpha, \beta, \gamma - 1)x = t(\alpha, \beta, \gamma) \), while if \( \gamma \in N_{\alpha\beta}(x), t(\alpha, \beta, \gamma)x = t(\alpha, \beta, \gamma - 1) \).

The set \( T' = \{ t(\alpha, \beta, \gamma)(\alpha, \beta) \in A \times B, 0 \leq \gamma \leq n \} \) is a finite subset of \( T \). Now choose and label any subset \( \{s(\alpha, \beta, \gamma)(\alpha, \beta) \in A \times B, 0 \leq \gamma \leq n \} \) of \( S \).
in one-to-one correspondence with $T'$, so that the correspondence $t(\alpha, \beta, \gamma) \leftrightarrow s(\alpha, \beta, \gamma)$ preserves order. Since multiplication by $x$ provides a one-to-one order preserving correspondence such that

\[
t(\alpha, \beta, \gamma - 1) \mapsto t(\alpha, \beta, \gamma) \quad \text{for } \gamma \in P_{\alpha \beta}(x),
\]

\[
t(\alpha, \beta, \gamma) \mapsto t(\alpha, \beta, \gamma - 1) \quad \text{for } \gamma \in N_{\alpha \beta}(x),
\]

it follows that the correspondence

\[
s(\alpha, \beta, \gamma - 1) \mapsto s(\alpha, \beta, \gamma) \quad \text{for } \gamma \in P_{\alpha \beta}(x),
\]

\[
s(\alpha, \beta, \gamma) \mapsto s(\alpha, \beta, \gamma - 1) \quad \text{for } \gamma \in N_{\alpha \beta}(x)
\]

must also be one-to-one and order preserving. As $H$ is o-2-transitive on $S$, $H$ is also o-n-transitive, and hence there exists $h(x) \in H$ such that

\[
s(\alpha, \beta, y - 1)h(x) = s(\alpha, \beta, y) \quad \text{for } y \in P_{\alpha \beta}(x),
\]

\[
s(\alpha, \beta, y)h(x) = s(\alpha, \beta, y - 1) \quad \text{for } y \in N_{\alpha \beta}(x).
\]

Since $t = t(\alpha, \beta, 0)$ for each $(\alpha, \beta)$, we may let $s = s(\alpha, \beta, 0)$. Then substituting $x \mapsto h(x)$ (and $x^{-1} \mapsto h(x^{-1}) = (h(x))^{-1}$, $e \mapsto e$), we have for each $(\alpha, \beta) \in A \times B$,

\[
s \prod_{\Gamma} h_{\alpha \beta y} = s(\alpha, \beta, n).
\]

Since $tw \neq t$,

\[
t = t \forall_{A} \land_{B} \prod_{\Gamma} x_{\alpha \beta y} = \forall_{A} \land_{B} t \prod_{\Gamma} x_{\alpha \beta y} = \forall_{A} \land_{B} t(\alpha, \beta, n),
\]

where on the right side of the equation, the lattice operations are taken on the finite chain $\{t(\alpha, \beta, n)\}$, which is in one-to-one order preserving correspondence with the chain $\{s(\alpha, \beta, n)\}$. It follows that

\[
s \forall_{A} \land_{B} \prod_{\Gamma} h_{\alpha \beta y} = \forall_{A} \land_{B} s \prod_{\Gamma} h_{\alpha \beta y} = \forall_{A} \land_{B} s(\alpha, \beta n) \neq s.
\]

Hence $\forall_{A} \land_{B} \prod_{\Gamma} h_{\alpha \beta y} \neq e$ in $H$, and $H$ fails to satisfy the identical relation $w = e$, proving Lemma 3.

Now to prove the Theorem, let $G$ be an $l$-group which is not normal valued and let $w = e$ be an identical relation which is not satisfied by every $l$-group. By Lemma 2, any variety containing $G$ must contain an $l$-group $H$ which is o-2-transitive on some totally ordered set. By Lemma 3, $H$ fails to satisfy $w = e$, and hence so does $G$.

**Corollary.** The variety of normal valued $l$-groups is the largest proper variety of $l$-groups and contains every other proper variety of $l$-groups.

In [2], Martinez showed that the variety $\mathcal{E}$ of all $l$-groups is finitely join irreducible in the lattice of varieties of $l$-groups. But a much stronger statement is now obvious.

**Corollary.** The variety $\mathcal{E}$ of all $l$-groups is completely join irreducible in the lattice of varieties of $l$-groups.
Corollary (Martinez [2]). The variety of normal valued $l$-groups is idempotent.

For a proof, it suffices to produce an $l$-group which is not an extension of a normal valued $l$-group by a normal valued $l$-group. The $l$-group $B(\mathbb{R})$ of all order-preserving permutations of the real line $\mathbb{R}$ having bounded support is $l$-simple [1] and not normal valued, and so serves this purpose.

The free $l$-group on a countable set is generic in the sense that it generates the variety of all $l$-groups. A much more tractable example is given by

Corollary (to Lemma 3). The $l$-group $A(\mathbb{R})$ of all order-preserving permutations of the real line is generic—if $A(\mathbb{R})$ satisfies a certain identical relation, every $l$-group satisfies that relation.

References


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