PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 57, Number 1, May 1976

# SOLVABILITY OF CONVOLUTION EQUATIONS IN $\mathcal{K}_1'$

S. SZNAJDER AND Z. ZIELEZNY

ABSTRACT. Let $S$ be a convolution operator in the space $\mathcal{K}_1'$ of distributions of exponential growth. A condition on $S$ introduced by O. von Grudzinski is proved to be equivalent to $S * \mathcal{K}_1' = \mathcal{K}_1'$.

This paper is motivated by a recent result of O. von Grudzinski [2], who characterized convolution operators in $\mathcal{D}'$ having fundamental solutions of exponential growth in $R^n$. Convolution operators in the space $\mathcal{D}'$ of all distributions in $R^n$ are distributions with compact support, i.e. in $\mathcal{E}'$. We state the main part of Grudzinski's theorem (see [2, Theorem 1.1]) in a form suitable for our purpose, using the space $\mathcal{K}_1'$ of distributions of exponential growth introduced by M. Hasumi [3].

Let $S$ be a distribution in $\mathcal{E}'$ and $\hat{S}$ its Fourier transform. The following conditions are equivalent:

(a) There exist positive constants $N$, $r$, $C$ such that

$$\sup_{s \in C^n, |s| \leqslant r} |\hat{S}(\xi + s)| \geqslant \frac{C}{(1 + |\xi|)^N}, \quad \xi \in R^n;$$

(b) $S$ has a fundamental solution in $\mathcal{K}_1'$.

We recall that a distribution $E \in \mathcal{D}'$ is a fundamental solution for $S \in \mathcal{E}'$ if $S * E = \delta$ where $*$ denotes the convolution and $\delta$ the Dirac measure at the origin.

We now ask the question of solvability of convolution equations in $\mathcal{K}_1'$. Let $\mathcal{O}_C'(\mathcal{K}_1' : \mathcal{K}_1')$ be the space of convolution operators in $\mathcal{K}_1'$ (see [3] or [5]). Under what conditions on $S \in \mathcal{O}_C'(\mathcal{K}_1' : \mathcal{K}_1')$ is $S * \mathcal{K}_1' = \mathcal{K}_1'$? The last equality means that the mapping $u \to S * u$ maps $\mathcal{K}_1'$ onto $\mathcal{K}_1'$.

THEOREM. *If $S$ is a distribution in $\mathcal{O}_C'(\mathcal{K}_1' : \mathcal{K}_1')$ then each of the conditions* (a) *and* (b) *is equivalent to* (c) $S * \mathcal{K}_1' = \mathcal{K}_1'$.

Before presenting the proof we recall the basic facts about the spaces $\mathcal{K}_1'$ and $\mathcal{O}_C'(\mathcal{K}_1' : \mathcal{K}_1')$; the proofs can be found in [3] or [5].

Let $\mathcal{K}_1$ be the space of all $C^\infty$-functions $\varphi$ in $R^n$ such that

$$\nu_k(\varphi) = \sup_{x \in R^n, |\alpha| \leqslant k} \gamma_k(x) |D^\alpha \varphi(x)| < \infty, \qquad k = 0, 1, \ldots,$$

Received by the editors July 7, 1975.
*AMS (MOS) subject classifications* (1970). Primary 45E10; Secondary 46F99.

where $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j = i^{-1}\partial/\partial x_j$ and $\gamma_k(x) = \prod_{j=1}^n (e^{kx_j} + e^{-kx_j})$. The topology in $\mathcal{K}_1$ is defined by the seminorms $v_k$. Then $\mathcal{K}_1$ is a Fréchet space.

The dual $\mathcal{K}_1'$ of $\mathcal{K}_1$ is the space of distributions of exponential growth. A distribution $u \in \mathcal{D}'$ is in $\mathcal{K}_1'$ if and only if there exists a multi-index $\beta$, an integer $k \geqslant 0$ and a bounded, continuous function $f$ in $R^n$ such that $u = D^\beta(\gamma_k f)$. $\mathcal{K}_1'$ is endowed with the topology of uniform convergence on all bounded sets in $\mathcal{K}_1$.

If $u \in \mathcal{K}_1'$ and $\varphi \in \mathcal{K}_1$, then the convolution $u * \varphi$ is a $C^\infty$-function defined by

$$u * \varphi(x) = \langle u_y, \varphi(x-y) \rangle$$

where $\langle u, \varphi \rangle = u(\varphi)$.

More generally, the space $\mathcal{O}_C'(\mathcal{K}_1' : \mathcal{K}_1')$ of convolution operators in $\mathcal{K}_1'$ consists of distributions $S \in \mathcal{K}_1'$ satisfying one of the equivalent conditions:

(i) The products $\gamma_k S$, $k = 0, 1, \dots$, are tempered distributions.

(ii) Given any $k = 0, 1, \dots$, $S$ can be represented in the form $S = \sum_{|\alpha| \leqslant m} D^\alpha f_\alpha$, where $f_\alpha$, $|\alpha| \leqslant m$, are continuous functions in $R^n$ such that

$$f_\alpha(x) = O(1/\gamma_k(x)) \quad \text{as } |x| \to \infty.$$

(iii) For every $\varphi \in \mathcal{K}_1$, $S * \varphi$ is in $\mathcal{K}_1$. Moreover, the mapping $\varphi \to S * \varphi$ of $\mathcal{K}_1$ into $\mathcal{K}_1$ is continuous.

If $S \in \mathcal{O}_C'(\mathcal{K}_1' : \mathcal{K}_1')$ and $\check{S}$ is obtained from $S$ by symmetry with respect to the origin, i.e. $\langle \check{S}, \varphi \rangle = \langle S_x, \varphi(-x) \rangle$, $\varphi \in \mathcal{K}_1$, then $\check{S}$ is also in $\mathcal{O}_C'(\mathcal{K}_1' : \mathcal{K}_1')$. The convolution of $S$ with $u \in \mathcal{K}_1'$ is then defined by

$$\langle S * u, \varphi \rangle = \langle u, \check{S} * \varphi \rangle, \qquad \varphi \in \mathcal{K}_1. \tag{1}$$

For $\varphi \in \mathcal{K}_1$, the Fourier transform

$$\hat{\varphi}(\xi) = \int_{R^n} e^{-i\langle \xi, x \rangle} \varphi(x)\, dx$$

can be continued in $C^n$ as an entire function such that

$$w_k(\hat{\varphi}) = \sup_{\zeta \in C^n, |\operatorname{Im} \zeta| \leqslant k} (1 + |\zeta|)^k |\hat{\varphi}(\zeta)| < \infty, \qquad k = 0, 1, \dots. \tag{2}$$

If $K_1$ is the space of all entire functions with property (2) and the topology in $K_1$ is defined by the seminorms $w_k$, then the Fourier transform is an isomorphism of $\mathcal{K}_1$ onto $K_1$.

The dual $\mathcal{K}_1'$ of $K_1$ is the space of Fourier transforms of distributions in $\mathcal{K}_1'$. For $u \in \mathcal{K}_1'$, the Fourier transform $\hat{u}$ is defined by the Parseval formula

$$\langle \hat{u}, \hat{\varphi} \rangle = (2\pi)^n \langle u_x, \varphi(-x) \rangle.$$

The Fourier transform $\hat{S}$ of a distribution $S \in \mathcal{O}_C'(\mathcal{K}_1' : \mathcal{K}_1')$ is a function which can be continued in $C^n$ as an entire function with the following property: for every $k = 0, 1, \dots$, there exists $l = 0, 1, \dots$, such that

$$\sup_{\zeta \in C^n, |\operatorname{Im} \zeta| \leqslant k} |\hat{S}(\zeta)|(1 + |\zeta|)^{-l} < \infty. \tag{3}$$

Also, if $S \in \mathcal{O}_C'(\mathcal{K}_1' : \mathcal{K}_1')$ and $u \in \mathcal{K}_1'$, we have the formula

$$\widehat{S * u} = \hat{S}\hat{u},$$

where the product on the right-hand side is defined by $\langle \hat{S}\hat{u}, \chi \rangle = \langle \hat{u}, \hat{S}\chi \rangle$, $\chi \in K_1$.

In the proof of our theorem we shall make use of the following lemma of L. Hörmander (see [4, Lemma 3.2]):

*If $F$, $G$ and $F/G$ are entire functions and $\rho$ is an arbitrary positive number, then*

$$|F(\zeta)/G(\zeta)| \leqslant \sup_{|\zeta - s| < 4\rho} |F(s)| \sup_{|\zeta - s| < 4\rho} |G(s)| \Big/ \Big( \sup_{|\zeta - s| < \rho} |G(s)| \Big)^2,$$

*where $\zeta, s \in C^n$.*

PROOF OF THE THEOREM. It is obvious that (c) $\Rightarrow$ (b). The implication (b) $\Rightarrow$ (a) was proved in [2] for $S \in \mathcal{E}'$. In the more general case where $S \in \mathcal{O}_C'(\mathcal{K}_1' : \mathcal{K}_1')$ the proof of this implication needs only minor modifications and therefore we leave it out. Thus it remains to show that (a) $\Rightarrow$ (c).

Let $S$ be a distribution in $\mathcal{O}_C'(\mathcal{K}_1' : \mathcal{K}_1')$ satisfying condition (a) and let $T = \check{S}$; in that case $T$ also satisfies condition (a). We consider the mapping $S *: u \to S * u$ of $\mathcal{K}_1'$ into $\mathcal{K}_1'$. By (1), $S *$ is the transpose of the mapping $T *: \varphi \to T * \varphi$ of $\mathcal{K}_1$ into $\mathcal{K}_1$. In order to prove (c) it suffices to show that $T *$ is an isomorphism of $\mathcal{K}_1$ onto $T * \mathcal{K}_1$ (see [1, Corollary, p. 92]).

By what we have said before, the mapping $T *$ is continuous. Also, using Fourier transforms it is easy to see that $T *$ is injective. We now prove that the inverse of $T *$, i.e. the mapping $T * \varphi \to \varphi$, is continuous. Since the Fourier transformation is an isomorphism, it suffices to prove the equivalent statement that the mapping $\hat{T}\hat{\varphi} \to \hat{\varphi}$ is continuous.

Suppose that $\hat{T}\hat{\varphi} = \hat{\psi}$ where $\hat{\varphi}, \hat{\psi} \in K_1$ and $\hat{T}$ is an entire function satisfying condition (a). We pick an integer $k \geqslant 0$ arbitrarily and assume that $\zeta = \xi + i\eta$ is in the horizontal strip $|\eta| \leqslant k$. Applying to the functions $\hat{\psi}$, $\hat{T}$ (and $\hat{\psi}/\hat{T} = \hat{\varphi}$) Hörmander's Lemma with $\rho = k + r$, we obtain

$$|\hat{\varphi}(\zeta)| \leqslant \sup_{|\zeta - s| < 4(k+r)} |\hat{\psi}(s)| \sup_{|\zeta - s| < 4(k+r)} |\hat{T}(s)| \Big/ \Big( \sup_{|\zeta - s| < k+r} |T(s)| \Big)^2. \tag{4}$$

But

$$\begin{aligned} \sup_{|\zeta - s| < k+r} |\hat{T}(s)| &= \sup_{|s| < k+r} |\hat{T}(\zeta + s)| \geqslant \sup_{|s| < r} |\hat{T}(\xi + s)| \\ &\geqslant C/(1 + |\xi|)^N \geqslant C/(1 + |\zeta|)^N, \end{aligned} \tag{5}$$

where we made use of condition (a).

On the other hand, since $|\eta| \leqslant k$, there exist constants $N', C' > 0$ such that

$$\sup_{|\zeta - s| < 4(k+r)} |\hat{T}(s)| = \sup_{|s| < 4(k+r)} |\hat{T}(\zeta + s)| \leqslant C'(1 + |\zeta|)^{N'}, \tag{6}$$

in view of (3).

Now combining (4) with (5) and (6) we obtain

$$|\hat{\varphi}(\zeta)| \leqslant C''(1 + |\zeta|)^{2N+N'} \sup_{|\zeta - s| < 4(k+r)} |\hat{\psi}(\zeta)|,$$

where $C''$ is a constant. Hence it follows that

$$w_k(\hat{\varphi}) \leqslant C^* w_l(\hat{\psi}),$$

where $C^*$ is another constant (depending only on $T$ and $k$) and $l$ is an integer $\geqslant \max\{5k + 4r, k + 2N + N'\}$. This proves the continuity of the mapping $\hat{T}\hat{\varphi} = \hat{\psi} \to \hat{\varphi}$ and consequently the implication (a) $\Rightarrow$ (c).

## References


1. J. Dieudonné and L. Schwartz, *La dualité dans les espaces* ($\mathfrak{F}$) *et* ($\mathfrak{LF}$), Ann. Inst. Fourier (Grenoble) **1** (1949), 61–101 (1950). MR **12**, 417.
2. O. von Grudzinski, *Über Fundamentallösungen von Convolutoren und von Differential-Differenzen-Operatoren mit knostanten Koeffizienten*, Dissertation, Kiel, 1974.
3. M. Hasumi, *Note on the n-dimensional tempered ultra-distributions*, Tôhoku Math. J. (2) **13** (1961), 94–104. MR **24** #A1607.
4. L. Hörmander, *On the range of convolution operators*, Ann. of Math. (2) **76** (1962), 148–170. MR **25** #5379.
5. Z. Zieleźny, *On the space of convolution operators in* $\mathcal{K}_1'$, Studia Math. **31** (1968), 111–124. MR **40** #1772.



Department of Mathematics, State University of New York at Buffalo, Amherst, New York 14226 (Current address of Z. Zielezny)

*Current address* (S. Sznajder): Department of Mathematics, University of Copenhagen, 2100 Copenhagan, Denmark