SOLVABILITY OF CONVOLUTION EQUATIONS IN $\mathcal{K}'_1$

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Abstract. Let $S$ be a convolution operator in the space $\mathcal{K}'_1$ of distributions of exponential growth. A condition on $S$ introduced by O. von Grudzinski is proved to be equivalent to $S \ast \mathcal{K}'_1 = \mathcal{K}'_1$.

This paper is motivated by a recent result of O. von Grudzinski [2], who characterized convolution operators in $\mathcal{B}'$ having fundamental solutions of exponential growth in $\mathbb{R}^n$. Convolution operators in the space $\mathcal{B}'$ of all distributions in $\mathbb{R}^n$ are distributions with compact support, i.e. in $\mathcal{B}'$. We state the main part of Grudzinski's theorem (see [2, Theorem 1.1]) in a form suitable for our purpose, using the space $\mathcal{K}'_1$ of distributions of exponential growth introduced by M. Hasumi [3].

Let $S$ be a distribution in $\mathcal{B}'$ and $\hat{S}$ its Fourier transform. The following conditions are equivalent:

(a) There exist positive constants $N$, $r$, $C$ such that

$$\sup_{s \in \mathbb{C}^n, |s| \leq r} |\hat{S}(\xi + s)| \geq \frac{C}{(1 + |\xi|)^N}, \quad \xi \in \mathbb{R}^n;$$

(b) $S$ has a fundamental solution in $\mathcal{K}'_1$.

We recall that a distribution $E \in \mathcal{B}'$ is a fundamental solution for $S \in \mathcal{B}'$ if $S \ast E = \delta$ where $\ast$ denotes the convolution and $\delta$ the Dirac measure at the origin.

We now ask the question of solvability of convolution equations in $\mathcal{K}'_1$. Let $\mathcal{O}_C(\mathcal{K}'_1 : \mathcal{K}'_1)$ be the space of convolution operators in $\mathcal{K}'_1$ (see [3] or [5]). Under what conditions on $S \in \mathcal{O}_C(\mathcal{K}'_1 : \mathcal{K}'_1)$ is $S \ast \mathcal{K}'_1 = \mathcal{K}'_1$? The last equality means that the mapping $u \mapsto S \ast u$ maps $\mathcal{K}'_1$ onto $\mathcal{K}'_1$.

Theorem. If $S$ is a distribution in $\mathcal{O}_C(\mathcal{K}'_1 : \mathcal{K}'_1)$ then each of the conditions (a) and (b) is equivalent to (c) $S \ast \mathcal{K}'_1 = \mathcal{K}'_1$.

Before presenting the proof we recall the basic facts about the spaces $\mathcal{K}'_1$ and $\mathcal{O}_C(\mathcal{K}'_1 : \mathcal{K}'_1)$; the proofs can be found in [3] or [5].

Let $\mathcal{K}'_1$ be the space of all $C^\infty$-functions $\varphi$ in $\mathbb{R}^n$ such that

$$v_k(\varphi) = \sup_{x \in \mathbb{R}^n, |x| \leq k} \gamma_k(x)|D^a \varphi(x)| < \infty, \quad k = 0, 1, \ldots,$$

Received by the editors July 7, 1975.


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where \( D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \) \( D_j = i^{-1} \partial / \partial x_j \) and \( \gamma_k(x) = \prod_{j=1}^n (e^{kx_j} + e^{-kx_j}). \) The topology in \( \mathcal{K}_1 \) is defined by the seminorms \( \nu_k. \) Then \( \mathcal{K}_1 \) is a Fréchet space.

The dual \( \mathcal{K}_1' \) of \( \mathcal{K}_1 \) is the space of distributions of exponential growth. A distribution \( u \in \mathcal{D}' \) is in \( \mathcal{K}_1' \) if and only if there exists a multi-index \( \beta, \) an integer \( k > 0 \) and a bounded, continuous function \( f \) in \( \mathbb{R}^n \) such that \( u = D^\beta(\gamma_k f). \) \( \mathcal{K}_1' \) is endowed with the topology of uniform convergence on all bounded sets in \( \mathcal{K}_1. \)

If \( u \in \mathcal{K}_1' \) and \( \varphi \in \mathcal{K}_1, \) then the convolution \( u \ast \varphi \) is a \( C^\infty \)-function defined by

\[
\langle u \ast \varphi, \varphi \rangle = \langle u, \varphi(x - y) \rangle
\]

where \( \langle u, \varphi \rangle = u(\varphi). \)

More generally, the space \( \mathcal{E}_c(\mathcal{K}_1' : \mathcal{K}_1') \) of convolution operators in \( \mathcal{K}_1' \) consists of distributions \( S \in \mathcal{K}_1' \) satisfying one of the equivalent conditions:

(i) The products \( \gamma_k S, k = 0, 1, \ldots, \) are tempered distributions.

(ii) Given any \( k = 0, 1, \ldots, \) \( S \) can be represented in the form \( S = \sum |a| \leq m D^a f_a, \) where \( f_a, |a| \leq m, \) are continuous functions in \( \mathbb{R}^n \) such that

\[
f_a(x) = O(1/\gamma_k(x)) \quad \text{as} \quad |x| \to \infty.
\]

(iii) For every \( \varphi \in \mathcal{K}_1, \) \( S \ast \varphi \) is in \( \mathcal{K}_1. \) Moreover, the mapping \( \varphi \to S \ast \varphi \) of \( \mathcal{K}_1 \) into \( \mathcal{K}_1 \) is continuous.

If \( S \in \mathcal{E}_c(\mathcal{K}_1' : \mathcal{K}_1') \) and \( \tilde{S} \) is obtained from \( S \) by symmetry with respect to the origin, i.e. \( \langle \tilde{S}, \varphi \rangle = \langle S_x, \varphi(-x) \rangle, \) \( \varphi \in \mathcal{K}_1, \) then \( \tilde{S} \) is also in \( \mathcal{E}_c(\mathcal{K}_1' : \mathcal{K}_1'). \) The convolution of \( S \) with \( u \in \mathcal{K}_1' \) is then defined by

\[
\langle S \ast u, \varphi \rangle = \langle u, \tilde{S} \ast \varphi \rangle, \quad \varphi \in \mathcal{K}_1.
\]

For \( \varphi \in \mathcal{K}_1, \) the Fourier transform

\[
\hat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle \xi, x \rangle} \varphi(x) \, dx
\]

can be continued in \( C^n \) as an entire function such that

\[
w_k(\hat{\varphi}) = \sup_{\xi \in C^n, |\text{Im} \xi| \leq k} (1 + |\xi|)^k |\hat{\varphi}(\xi)| < \infty, \quad k = 0, 1, \ldots
\]

If \( K_1 \) is the space of all entire functions with property (2) and the topology in \( K_1 \) is defined by the seminorms \( w_k, \) then the Fourier transform is an isomorphism of \( \mathcal{K}_1 \) onto \( K_1. \)

The dual \( \mathcal{K}_1' \) of \( K_1 \) is the space of Fourier transforms of distributions in \( \mathcal{K}_1. \) For \( u \in \mathcal{K}_1', \) the Fourier transform \( \hat{u} \) is defined by the Parseval formula

\[
\langle \hat{u}, \varphi \rangle = (2\pi)^n \langle u_x, \varphi(-x) \rangle.
\]

The Fourier transform \( \tilde{S} \) of a distribution \( S \in \mathcal{E}_c(\mathcal{K}_1' : \mathcal{K}_1') \) is a function which can be continued in \( C^n \) as an entire function with the following property: for every \( k = 0, 1, \ldots, \) there exists \( l = 0, 1, \ldots, \) such that

\[
\sup_{\xi \in C^n, |\text{Im} \xi| \leq k} |\tilde{S}(\xi)|(1 + |\xi|)^{-l} < \infty.
\]
Also, if \( S \in \mathcal{E}_c(\mathcal{K}'_1 : \mathcal{K}_1) \) and \( u \in \mathcal{K}'_1 \), we have the formula
\[
S \hat{*} u = \hat{S}u,
\]
where the product on the right-hand side is defined by \( \langle \hat{S}u, \chi \rangle = \langle \hat{u}, \hat{S}\chi \rangle \), \( \chi \in \mathcal{K}_1 \).

In the proof of our theorem we shall make use of the following lemma of L. Hörmander (see [4, Lemma 3.2]):

If \( F, G \) and \( F/G \) are entire functions and \( \rho \) is an arbitrary positive number, then
\[
\left| \frac{F(\xi)}{G(\xi)} \right| \leq \sup_{|\xi-s|<4\rho} |F(s)| \sup_{|\xi-s|<4\rho} |G(s)| \left( \frac{\sup_{|\xi|<\rho} |G(s)|}{\sup_{|\xi|<\rho} |F(s)|} \right)^2,
\]
where \( \xi, s \in \mathbb{C}^n \).

Proof of the theorem. It is obvious that \( (c) \Rightarrow (b) \). The implication \( (b) \Rightarrow (a) \) was proved in [2] for \( S \in \mathcal{E}' \). In the more general case where \( S \in \mathcal{E}_c(\mathcal{K}'_1 : \mathcal{K}_1) \) the proof of this implication needs only minor modifications and therefore we leave it out. Thus it remains to show that \( (a) \Rightarrow (c) \).

Let \( S \) be a distribution in \( \mathcal{E}_c(\mathcal{K}'_1 : \mathcal{K}_1) \) satisfying condition \( (a) \) and let \( T = \hat{S} \); in that case \( T \) also satisfies condition \( (a) \). We consider the mapping \( S \hat{*}: u \to S \hat{*} u \) of \( \mathcal{K}'_1 \) into \( \mathcal{K}_1 \). By (1), \( S \hat{*} \) is the transpose of the mapping \( T \hat{*}: \varphi \to T \hat{*} \varphi \) of \( \mathcal{K}_1 \) into \( \mathcal{K}_1 \). In order to prove \( (c) \) it suffices to show that \( T \hat{*} \) is an isomorphism of \( \mathcal{K}_1 \) onto \( T \hat{*} \mathcal{K}_1 \) (see [1, Corollary, p. 92]).

By what we have said before, the mapping \( T \hat{*} \) is continuous. Also, using Fourier transforms it is easy to see that \( T \hat{*} \) is injective. We now prove that the inverse of \( T \hat{*} \), i.e. the mapping \( T \hat{*} \varphi \to \varphi \), is continuous. Since the Fourier transformation is an isomorphism, it suffices to prove the equivalent statement that the mapping \( \hat{T}\varphi \to \hat{\varphi} \) is continuous.

Suppose that \( \hat{T}\varphi = \hat{\psi} \) where \( \hat{\varphi}, \hat{\psi} \in \mathcal{K}_1 \) and \( \hat{T} \) is an entire function satisfying condition \( (a) \). We pick an integer \( k \geq 0 \) arbitrarily and assume that \( \xi = \xi + i\eta \) is in the horizontal strip \( |\eta| < k \). Applying to the functions \( \hat{\varphi}, \hat{T} \) (and \( \hat{\psi}/\hat{T} = \hat{\varphi} \)) Hörmander’s Lemma with \( \rho = k + r \), we obtain

\[
|\hat{\varphi}(\xi)| \leq \sup_{|\xi-s|<4(k+r)} |\hat{\psi}(s)| \sup_{|\xi-s|<4(k+r)} |\hat{T}(s)| \left( \frac{\sup_{|\xi|<\rho} |\hat{T}(s)|}{\sup_{|\xi|<\rho} |\hat{\psi}(s)|} \right)^2.
\]

But
\[
\sup_{|\xi-s|<k+r} |\hat{T}(s)| = \sup_{|s|<k+r} |\hat{T}(\xi + s)| \geq \sup_{|s|<r} |\hat{T}(\xi + s)|
\]
\[
\geq C/(1 + |\xi|)^N \geq C/(1 + |\xi|)^N,
\]
where we made use of condition \( (a) \).

On the other hand, since \( |\eta| \leq k \), there exist constants \( N', C' > 0 \) such that
\[
\sup_{|\xi-s|<4(k+r)} |\hat{T}(s)| = \sup_{|s|<4(k+r)} |\hat{T}(\xi + s)| \leq C'(1 + |\xi|)^{N'},
\]
in view of (3).

Now combining (4) with (5) and (6) we obtain
\[ |\hat{\varphi}(\xi)| \leq C^\ast (1 + |\xi|)^{2N + N'} \sup_{|\xi - s| < 4(k + r)} |\hat{\psi}(\xi)|, \]

where \( C^\ast \) is a constant. Hence it follows that

\[ w_k(\hat{\varphi}) \leq C^\ast w_l(\hat{\psi}), \]

where \( C^\ast \) is another constant (depending only on \( T \) and \( k \)) and \( l \) is an integer \( \geq \max\{5k + 4r, k + 2N + N'\} \). This proves the continuity of the mapping \( T\hat{\varphi} = \hat{\psi} \rightarrow \hat{\varphi} \) and consequently the implication \( (a) \Rightarrow (c) \).

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