RELATING GROUP TOPOLOGIES BY THEIR CONTINUOUS POINTS

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ABSTRACT. Let \( x \) be a point in a topological group \( G \), and for each integer \( n \), let \( (1/n)x \) be the set \( \{ y : ny = x \} \) in \( G \). Then I call \( x \) a continuous point if for positive integers \( n \), the subsets \( (1/n)x \) are nonvoid and eventually intersect each neighbourhood of the identity \( 0 \). I prove the following result and from it three corollaries. Let \( G \) be a divisible abelian group such that \( (1/n)0 = \{ 0 \} \) for some integer \( n > 2 \). Suppose there are two group topologies \( \tau_1 \) and \( \tau_2 \) defined on \( G \) and that \( G \) is \( \tau_2 \)-locally compact and \( \sigma \)-compact, and define \( \omega_2 \) to be the outer measure derived from the Haar measure \( \mu_2 \) on \( (G, \tau_2) \). Also suppose that the ratio of the \( \tau_2 \)-measure of \( \{ nx : x \in A \} \) to the \( \tau_2 \)-measure of \( A \), for any \( \tau_2 \)-Borel-measurable set \( A \) (the ratio is the same for any such \( A \) with finite measure), does not exceed 1. Then for each \( \tau_2 \)-Borel-measurable set \( A \) with nonvoid \( \tau_1 \)-interior, \( \mu_2(A) > \omega_2(W_1) \), where \( W_1 \) is the subgroup of all points in \( G \) which are \( \tau_1 \)-continuous.

The study of compact group topologies for the real line gave rise to the rather interesting questions posed by D. N Hawley [1] and answered by me for \( R^N \) [4]. I propose to present now a generalization of the proofs in [4], something which supplies the basis for the study of what I call the continuous points in a topological group (see [5]). This work forms part of a Ph.D. thesis submitted to La Trobe University in Melbourne, Australia, and was done under the supervision of Dr. Graham Elton.

DEFINITIONS. Let \( G \) be a group (I write my groups additively) and \( A \) a subset of \( G \). It is possible to define two kinds of "\( n \)th-multiples" of the set \( A \):
\[
nA = \{ x_1 + x_2 + \cdots + x_n : x_1, x_2, \ldots, x_n \in A \},
\]
for \( n \) a positive integer, and
\[
o nA = \{ nx : x \in A \},
\]
for \( n \) any integer.

An element \( x \) of \( G \) is divisible (in \( G \)) if for each positive integer \( n \) there is a \( y \) in \( G \) satisfying \( x = ny \). If every element of \( G \) is divisible in \( G \), then \( G \) is said to be divisible. To avoid excess of writing, I put \( (1/n)x = \{ y : ny = x \} \), and for \( A \) a subset of \( G \), \( o (1/n)A = \{ y : ny \in A \} \).

Now consider \( G \) to be a topological group. I call a divisible element \( x \) of \( G \)

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a continuous point if the subsets \((1/n)x\), for positive integers \(n\), eventually intersect each neighbourhood of the identity. In other words, if \(A\) is a neighbourhood of the identity and \(x\) is a continuous point, there is a positive integer \(N\) such that for all \(n > N\), \((1/n)x \cap A \neq \emptyset\). I designate the collection of continuous points in \(G\) by \(W\), and if \(G\) is abelian, \(W\) is a subgroup.

Most of this work concerns groups which are divisible and abelian; these I call \(da\) groups for short. I am also concerned with the torsion-free property in that it involves this idea: \(G\) is uniquely \(n\)th-rooted if \(y_1\) and \(y_2\) in \(G\) are such that \(ny_1 = ny_2\), then \(y_1 = y_2\) (\(n\) is a positive integer). If \(G\) is an abelian group, then the uniquely \(n\)th-rooted property is equivalent to \(G\)'s containing no points, except the identity 0, whose \(n\)th-multiple is 0. (Note that if \(x\) in \(G\) is uniquely \(n\)th-rooted, \((1/n)x\) contains at most one point, and I take \((1/n)x\) to be that point.)

First I want to show that \(\circ^nA\) and \(\circ(1/n)A\) are Borel (-measurable) for a Borel set \(A\) and a positive integer \(n\).

**Lemma 1.** If \(G\) is a \(da\) \(\sigma\)-compact locally compact group, then, for each positive integer \(n\), the function \(f_n: x \rightarrow nx\), for all \(x\) in \(G\), is an open and continuous homomorphism of \(G\) onto \(G\).

**Proof.** That \(f_n\) is continuous follows simply from the definition of a topological group (see [3, p. 96, part A]), and the open property follows from (5.29) in [2, p. 42].

It is true then that, in any topological group, not only translates and inverses of Borel sets are again Borel, but also that the \(n\)th-multiples of Borel sets are Borel when the group is \(\sigma\)-compact, locally compact and uniquely \(n\)th-rooted.

To my main train of thought. I want to build up to the fact that for certain groups \(G\) with Haar measure \(\mu\), \(\mu(\circ^nA) \leq \mu(A)\) for a positive integer \(n\) and for all \(A\) in \(\mathcal{M}\), the \(\sigma\)-algebra of all Borel sets. To do this define \(\mu^n\) by \(\mu^n(A) = \mu(\circ^nA)\) for all \(A\) in \(\mathcal{M}\). If \(G\) is \(da\), uniquely \(n\)th-rooted, locally compact and \(\sigma\)-compact, then \(\mu^n\) is a Haar measure on \(G\); for instance

\[
\mu^n(x + A) = \mu(\circ^n(x + A)) = \mu(nx + \circ^nA) = \mu(\circ^nA) = \mu^n(A),
\]

for all \(x\) in \(G\) and \(A\) in \(\mathcal{M}\). But the Haar measure on \(G\) is essentially unique, and so there is a positive real \(c_n\) such that \(\mu^n = c_n\mu\). It can be shown that \(c_n\) is the product of integer powers of the prime factors of \(n\), but more important for this study, it can be shown that \(c_n\) is not dependent on the particular Haar measure chosen for the topology. If \(c_n < 1\), then \(\mu(\circ^nA) = \mu^n(A) < \mu(A)\) for all \(A\) in \(\mathcal{M}\), and this would be the case if, for instance, \(G\) contains a compact open subgroup. To summarize these results:

**Lemma 2.** Let \(G\) be a \(da\), uniquely \(n\)th-rooted, \(\sigma\)-compact, locally compact group with \(c_n < 1\) for some integer \(n > 2\). Then \(\mu(\circ^nA) < \mu(A)\) for all Borel sets \(A\) and for a Haar measure \(\mu\).

Now to my main result.

**Theorem 3.** Let \(G\) be a \(da\), uniquely \(n\)th-rooted group for some integer \(n > 2\). Suppose there are two group topologies \(\mathcal{E}_1\) and \(\mathcal{E}_2\) defined on \(G\), such that \((G,
\( \mathcal{A}_2 \) is locally compact, \( \sigma \)-compact and for it \( c_n \leq 1 \), and define \( \omega_2 \) to be the outer measure derived from the Haar measure \( \mu_2 \) on \((G, \mathcal{A}_2)\). Then for any \( \mathcal{A}_2 \)-Borel set \( A \) with \( \mathcal{A}_1 \)-interior containing 0,

\[
\omega_2(W_1) \leq \mu_2 \left( \bigcup_{v=0}^{\infty} n^v \left( \bigcap_{m=0}^{\infty} n^m A \right) \right) = \mu_2 \left( \bigcap_{m=0}^{\infty} n^m A \right) \leq \mu_2(A),
\]

where \( W_1 \) is the subgroup of all the points in \( G \) which are \( \mathcal{A}_1 \)-continuous.

PROOF. Let \( A \) be such an \( \mathcal{A}_2 \)-Borel set which contains the \( \mathcal{A}_1 \)-open neighbourhood \( B \) of 0. Then let \( \mathcal{K} = \bigcap_{m=0}^{\infty} n^m A \), and \( \mathcal{L} = \bigcup_{v=0}^{\infty} n^v \mathcal{K} ; \) from Lemma 1 it follows that \( \mathcal{K} \) and, hence, \( \mathcal{L} \) are \( \mathcal{A}_2 \)-Borel.

As each \( x \) in \( W_1 \) is an \( \mathcal{A}_1 \)-continuous point, the sets \( (1/u)x \), for positive integers \( u \), eventually intersect \( B \) and, hence, they eventually intersect \( A \). That is: there is a positive integer \( U \) such that for any \( u > U \), \( (1/u)x \cap A \neq \emptyset \). Take \( q \) an integer such that \( n^q > U \). Then for all \( m > 0 \), \( n^m n^q > U \) and \( (1/n^m n^q)x \cap A \neq \emptyset \). Now for any positive integers \( a \) and \( c \),

\[
(1/a) \left[ (1/c)x \right] = (1/a) \left\{ y \in G : cy = x \right\} = \{ z \in G : az = y \text{ and } cy = x \text{ for some } y \in G \}
\]

\[
= \{ z \in G : acz = x \} = (1/ac)x.
\]

Hence \( (1/n^m)((1/n^q)x) \cap A \neq \emptyset \) for all \( m > 0 \). This and the fact that \( G \) is uniquely \( n^q \)-th-rooted put \( (1/n^q)x \) in \( n^m A \), for all \( m > 0 \), and thus in \( K \). So \( x \) is in \( n^q \mathcal{K} \subseteq \mathcal{L} \), and \( W_1 \subseteq \mathcal{L} \).

I want to show that \( (n^m \mathcal{K})_{v=0}^{\infty} \) is an ordered chain of subsets, that is, \( n^v \mathcal{K} \subseteq n^{v+1} \mathcal{K} \) for any positive integer \( v \). If \( x \) is in \( n^v \mathcal{K} \), then \( (1/n^v)x \) is in \( K \) and \( (1/n^v)x \) is in \( n^m A \) for all \( m > 0 \). Hence, \( (1/n^v)x \) is an element of \( n(n^m A) = n^{m+1} A \) for all \( m > 0 \), and taking \( n \)-th-roots, \( (1/n)((1/n^v)x) \) is an element of \( n^m A \) for all \( m > 0 \). So \( (1/n^{v+1})x \) is in \( \bigcap_{m=0}^{\infty} n^m A = K \) and \( x \) in \( n^{v+1} \mathcal{K} \), making \( n^v \mathcal{K} \subseteq n^{v+1} \mathcal{K} \).

The fact that \( n^v \mathcal{K} \subseteq n^{v+1} \mathcal{K} \) for all \( v > 0 \) means that

\[
\mu_2 \left( \bigcup_{v=0}^{\infty} n^v \mathcal{K} \right) = \lim_{v \to \infty} \mu_2(n^v \mathcal{K}),
\]

which is less than \( \mu_2(K) \) by Lemma 2. Combining this with the facts that \( W_1 \subseteq \mathcal{L} \), \( \mathcal{K} \subseteq \mathcal{L} \), and \( A \supseteq \mathcal{K} \), gives \( \omega_2(W_1) \leq \mu_2(L) = \mu_2(K) \leq \mu_2(A) \), the required result.

There are three corollaries from this result, the second two of which form the basis of further work in this field (see [5]). The definition stated below arises from a generalization of the condition Hawley was interested in for the reals, and is used in Corollary 6.

Corollary 4. Let \( G, \mathcal{A}_1 \), and \( \mathcal{A}_2 \) be as in Theorem 3, except that for the second topology \( c_n < 1 \). Then \( W_1 \) is \( \mathcal{A}_2 \)-negligible if there is an \( \mathcal{A}_2 \)-Borel set with finite \( \mathcal{A}_2 \)-measure and having a nonvoid \( \mathcal{A}_1 \)-interior.

Corollary 5. Let \( G \) be a da, uniquely nth-rooted, nondiscrete topological group for some integer \( n > 2 \), and which is locally compact, \( \sigma \)-compact and has \( c_n < 1 \). Then the subgroup of continuous points in \( G \) is negligible.
Proof. Putting \( \mathcal{A}_1 = \mathcal{A}_2 \) in Theorem 3 implies that every open set has measure at least the outer measure of the subgroup of continuous points. But by regularity, since a point has zero measure, there are sets open in \( G \) with arbitrarily small measures.

Definition. Suppose there are two topologies \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) defined on some space \( X \). Then \( \mathcal{A}_2 \) is Hawley with respect to \( \mathcal{A}_1 \) if, given any \( \mathcal{A}_2 \)-Borel set, either it or its complement is dense in \((X, \mathcal{A}_1)\).

Corollary 6. Let \( G \) be a da, uniquely nth-rooted group for some integer \( n \geq 2 \). Suppose there are two group topologies \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) defined on \( G \), and \( \mathcal{A}_2 \) causes \( G \) to be compact. Then \( \mathcal{A}_2 \) is Hawley with respect to \( \mathcal{A}_1 \) if the subgroup of \( \mathcal{A}_1 \)-continuous points is not \( \mathcal{A}_2 \)-negligible.

Proof. In any compact and connected group, a nonnegligible subgroup has outer measure 1. Now by applying Theorem 3 with \( c_n = 1 \) to our present group, it can be seen that every \( \mathcal{A}_2 \)-Borel set with nonvoid \( \mathcal{A}_1 \)-interior must have \( \mathcal{A}_2 \)-measure 1. Thus the \( \mathcal{A}_2 \)-measure of \( G \) is two times what it should be if an \( \mathcal{A}_2 \)-Borel set and its complement are both not dense in \((G, \mathcal{A}_1)\).

If \( \mathcal{A}_2 \) is Hawley with respect to \( \mathcal{A}_1 \) for two topologies \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) on some space \( X \), then the only functions from \( X \) to a Hausdorff space both \( \mathcal{A}_1 \)-continuous and \( \mathcal{A}_2 \)-Borel-measurable are the constant functions. This can be proved in exactly the same way as Theorem 4 in [4]. However, it is not possible to remove the “\( \mathcal{A}_1 \)-continuous” and make it “\( \mathcal{A}_1 \)-Borel-measurable”, for if \((X, \mathcal{A}_1)\) and \((X, \mathcal{A}_2)\) are Hausdorff spaces and \( X \) contains two distinct points \( x \) and \( y \), the map \( f: X \to \{x, y\} \) defined by \( f(x) = x \) and \( f(z) = y \) if \( z \) is in \( \{x\}' \), is \( \mathcal{A}_1 \)- and \( \mathcal{A}_2 \)-Borel-measurable, but is not a constant function.

References

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