JOIN-PRINCIPAL ELEMENT LATTICES

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Abstract. Let \((\mathcal{L}, M)\) be a local Noether lattice. If the maximal element \(M\) is meet principal, it is well known and easily seen that every element of \(\mathcal{L}\) is meet principal. In this note, we obtain the corresponding result for \(M\) joinprincipal. We also consider join-principal elements generally under the assumption of the weak union condition and show, for example, that the square of a join-principal element is principal.

Throughout we assume that \(\mathcal{L}\) is a local Noether lattice with maximal element \(M\).

Theorem 1. If \(A \in \mathcal{L}\) is join-principal, then, for each \(n \geq 1\), \(A^n\) has a unique minimal basis in \(\mathcal{L}/((0 : A) \wedge A)\).

Proof. Use Theorem 1 of [2] to choose a minimal base \(E_1, \ldots, E_k\) for \(A\) such that \(E_i E_j = 0\) whenever \(i \neq j\).

Let \(F\) be any principal element \(\leq A^n \vee ((0 : A) \wedge A)\). Then by Lemma 1 of [3], there exist principal elements \(F_1, \ldots, F_{k+1}\) such that

\[
F \vee \left( \bigvee_{i \neq j} F_i E_i^n \right) = \bigvee_i F_i E_i^n
\]

for \(j = 1, \ldots, k + 1\), where \(E_{k+1} = I\) and \(F_{k+1} \leq (0 : A) \wedge A\).

Then \(FE_j = F_j E_j^{n+1}\) (\(1 \leq j \leq k\)), so

\[
FA = \bigvee_{1 \leq j \leq k} F_j E_j^{n+1} = \bigvee_{1 \leq j \leq k} F_j E_j^n A.
\]

Hence

\[
F \vee (0 : A) = \left( \bigvee_{1 \leq j \leq k} F_j E_j^n \right) \vee (0 : A).
\]

So, since principal elements are join-irreducible,

\[
F \vee ((0 : A) \wedge A) = F_j E_j^n \vee ((0 : A) \wedge A),
\]

for some \(j\). Hence either \(F \vee ((0 : A) \wedge A) = E_j^n \vee ((0 : A) \wedge A)\), or \(F \leq MA^n \vee ((0 : A) \wedge A)\) and \(F\) cannot be used in a minimal base for \(A^n\) in...
\( \mathcal{L}/(0 : A) \wedge A \). It follows that the only minimal bases for \( A^n \) in \( \mathcal{L}/((0 : A) \wedge A) \) are subsets of \( E^n_1 \vee ((0 : A) \wedge A), \ldots, E^n_k \vee ((0 : A) \wedge A) \). On the other hand, if \( E^n_j \leq E^n_1 \vee \cdots \vee E^n_j \vee \cdots \vee E^n_k \vee ((0 : A) \wedge A) \), then \( E^n_j A = E^n_{j+1} \leq (E^n_1 \vee \cdots \vee E^n_j \vee \cdots \vee E^n_k \vee ((0 : A) \wedge A)) E_j = 0 \). Hence \( A^n \) has the unique minimal base in \( \mathcal{L}/A \wedge (0 : A) \) consisting of those \( E^n_j \vee (A \wedge (0 : A)) \) with \( E^n_j A \neq 0 \). Q.E.D.

**Corollary 2.** If \( \mathcal{L} \) satisfies the weak union condition and if \( A \in \mathcal{L} \) is join-principal, then \( A = E \vee ((0 : A) \wedge A) \) for some principal element \( E \). In particular, \( A^2 = E^2 \) is principal.

**Proof.** If the weak union condition is satisfied, only principal elements have unique minimal bases. Since \( \mathcal{L}/A \wedge (0 : A) \) inherits the hypothesis from \( \mathcal{L} \), the result follows.

**Corollary 3.** Let \((R, M)\) be a local ring in which \( M \) is join-principal. Then, as an \( R \)-module, \( M \) is the direct sum of a cyclic \( R \)-module and an \( R/M \)-vector space.

In the general setting, we get the following

**Theorem 4.** If the maximal element \( M \) of \( \mathcal{L} \) is join-principal, then \( \mathcal{L}/(0: M) \) is distributive.

**Proof.** Let \( F \) be any principal element of \( \mathcal{L} \), \( F \leq 0 : M \). By Theorem 1 and Corollary 1 of [2], there exist principal elements \( E_1, \ldots, E_k \) in \( \mathcal{L} \) such that \( E_1 \vee (0 : M), \ldots, E_k \vee (0 : M) \) are an independent minimal base of \( M \) in \( \mathcal{L}/(0 : M) \). Choose \( n \) so that \( F \leq M^n \vee (0 : M) \) and \( F \leq M^{n+1} \vee (0 : M) \).

Since \( M \) is join-principal in \( \mathcal{L} \), it follows that \( F \vee (0 : M) = E^n_i \vee (0 : M) \) for some \( i \). Since the \( E_j \vee (0 : M) \) are independent, it follows that \( \mathcal{L}/(0 : M) \) is distributive. Q.E.D.

**Corollary 5.** If the maximal element \( M \) of \( \mathcal{L} \) is join-principal, then every element of \( \mathcal{L} \) is join-principal.

**Proof.** Let \( B \) and \( A \) be any elements of \( \mathcal{L} \). Since \( M \) is join-principal in \( \mathcal{L}/C \) for every \( C \), it suffices to show that \( BA : A = B \vee (0 : A) \).

Let \( E_1, \ldots, E_k \) be independent principal elements such that \( E_1 \vee (0 : M), \ldots, E_k \vee (0 : M) \) form the minimal base for \( M \) in \( \mathcal{L}/0 : M \). Then from the proof of Theorem 4, \( AB : A \) and \( B \vee (0 : A) \) are both joins of powers of the \( E_i \vee (0 : M) \) in \( \mathcal{L}/0 : M \).

Set \( B \vee (0 : A) = E^{s_1}_1 \vee \cdots \vee E^{s_k}_k \vee (0 : M) \), and assume \( E^n_i \leq BA : A \). Then \( E^n_i A \leq A(B \vee (0 : A)) = AE^{s_1}_1 \vee \cdots \vee AE^{s_k}_k \). Since \( E_j X \) is a power of \( E_j \) for every \( X \neq 1 \) and since the \( E_j \) are independent, it follows that \( E^n_i A \leq E^{s_i}_i A \). If \( n < s_i \), then \( A \leq E^{s_i-n}_i A \vee (0 : E^n_i) \). However, in this case, \( A \leq 0 : E^n_i \) and \( E^n_i \leq 0 : A \). If \( n \geq s_i \), then \( E^n_i \leq E^{s_i}_i \vee \cdots \vee E^{s_k}_k \). In either case, \( E^n_i \leq B \vee (0 : A) \). Hence \( BA : A \leq B \vee (0 : A) \). Q.E.D.

**References**

1. E. W. Johnson and J. P. Lediaev, *Structure of Noether lattices into join-principal maximal*

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