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JOIN-PRINCIPAL ELEMENT LATTICES

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ABSTRACT. Let $(\mathfrak{L}, M)$ be a local Noether lattice. If the maximal element M is meet principal, it is well known and easily seen that every element of $\mathfrak{L}$ is meet principal. In this note, we obtain the corresponding result for M join-principal. We also consider join-principal elements generally under the assumption of the weak union condition and show, for example, that the square of a join-principal element is principal.

Throughout we assume that $\mathfrak{L}$ is a local Noether lattice with maximal element M.

THEOREM 1. *If $A \in \mathfrak{L}$ is join-principal, then, for each $n \geqslant 1$, A^n has a unique minimal basis in $\mathfrak{L}/((0:A) \wedge A)$.*

PROOF. Use Theorem 1 of [2] to choose a minimal base $E_1, \ldots, E_k$ for A such that $E_i E_j = 0$ whenever $i \neq j$.

Let F be any principal element $\leqslant A^n \vee ((0:A) \wedge A)$. Then by Lemma 1 of [3], there exist principal elements $F_1, \ldots, F_{k+1}$ such that

$$F \vee \left(\bigvee_{i \neq j} F_i E_i^n\right) = \bigvee_i F_i E_i^n$$

for $j = 1, \ldots, k+1$, where $E_{k+1} = I$ and $F_{k+1} \leqslant (0:A) \wedge A$.

Then $FE_j = F_j E_j^{n+1}$ $(1 \leqslant j \leqslant k)$, so

$$FA = \bigvee_{1 \leqslant j \leqslant k} F_j E_j^{n+1} = \bigvee_{1 \leqslant j \leqslant k} F_j E_j^n A.$$

Hence

$$F \vee (0:A) = \left(\bigvee_{1 \leqslant j \leqslant k} F_j E_j^n\right) \vee (0:A).$$

So, since principal elements are join-irreducible,

$$F \vee ((0:A) \wedge A) = F_j E_j^n \vee ((0:A) \wedge A),$$

for some j. Hence either $F \vee ((0:A) \wedge A) = E_j^n \vee ((0:A) \wedge A)$, or $F \leqslant MA^n \vee ((0:A) \wedge A)$ and F cannot be used in a minimal base for A^n in

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$\mathfrak{L}/(0:A) \wedge A$. It follows that the only minimal bases for A^n in $\mathfrak{L}/((0:A) \wedge A)$ are subsets of $E_1^n \vee ((0:A) \wedge A), \ldots, E_k^n \vee ((0:A) \wedge A)$. On the other hand, if $E_j^n \leqslant E_1^n \vee \cdots \vee \hat{E}_j^n \vee \cdots \vee E_k^n \vee ((0:A) \wedge A)$, then $E_j^n A = E_j^{n+1} \leqslant (E_1^n \vee \cdots \vee \hat{E}_j^n \vee \cdots \vee E_k^n \vee ((0:A) \wedge A))E_j = 0$. Hence A^n has the unique minimal base in $\mathfrak{L}/A \wedge (0:A)$ consisting of those $E_j^n \vee (A \wedge (0:A))$ with $E_j^n A \neq 0$. Q.E.D.

COROLLARY 2. *If $\mathfrak{L}$ satisfies the weak union condition and if $A \in \mathfrak{L}$ is join-principal, then $A = E \vee ((0:A) \wedge A)$ for some principal element E. In particular, $A^2 = E^2$ is principal.*

PROOF. If the weak union condition is satisfied, only principal elements have unique minimal bases. Since $\mathfrak{L}/A \wedge (0:A)$ inherits the hypothesis from $\mathfrak{L}$, the result follows.

COROLLARY 3. *Let (R, M) be a local ring in which M is join-principal. Then, as an R-module, M is the direct sum of a cyclic R-module and an R/M-vector space.*

In the general setting, we get the following

THEOREM 4. *If the maximal element M of $\mathfrak{L}$ is join-principal, then $\mathfrak{L}/(0:M)$ is distributive.*

PROOF. Let F be any principal element of $\mathfrak{L}$, $F \nleqslant 0:M$. By Theorem 1 and Corollary 1 of [2], there exist principal elements $E_1, \ldots, E_k$ in $\mathfrak{L}$ such that $E_1 \vee (0:M), \ldots, E_k \vee (0:M)$ are an independent minimal base of M in $\mathfrak{L}/(0:M)$. Choose n so that $F \leqslant M^n \vee (0:M)$ and $F \nleqslant M^{n+1} \vee (0:M)$. Since M is join-principal in $\mathfrak{L}$, it follows that $F \vee (0:M) = E_i^n \vee (0:M)$ for some i. Since the $E_j \vee (0:M)$ are independent, it follows that $\mathfrak{L}/(0:M)$ is distributive. Q.E.D.

COROLLARY 5. *If the maximal element M of $\mathfrak{L}$ is join-principal, then every element of $\mathfrak{L}$ is join-principal.*

PROOF. Let B and A be any elements of $\mathfrak{L}$. Since M is join-principal in $\mathfrak{L}/C$ for every C, it suffices to show that $BA:A = B \vee (0:A)$.

Let $E_1, \ldots, E_k$ be independent principal elements such that $E_1 \vee (0:M), \ldots, E_k \vee (0:M)$ form the minimal base for M in $\mathfrak{L}/0:M$. Then from the proof of Theorem 4, $AB:A$ and $B \vee (0:A)$ are both joins of powers of the $E_i \vee (0:M)$ in $\mathfrak{L}/0:M$.

Set $B \vee (0:A) = E_1^{s_1} \vee \cdots \vee E_k^{s_k} \vee (0:M)$, and assume $E_i^n \leqslant BA:A$. Then $E_i^n A \leqslant A(B \vee (0:A)) = AE_1^{s_1} \vee \cdots \vee AE_k^{s_k}$. Since $E_j X$ is a power of E_j for every $X \neq I$ and since the E_j are independent, it follows that $E_i^n A \leqslant E_i^{s_i} A$. If $n < s_i$, then $A \leqslant E_i^{s_i - n} A \vee (0:E_i^n)$. However, in this case, $A \leqslant 0:E_i^n$ and $E_i^n \leqslant 0:A$. If $n \geqslant s_i$, then $E_i^n \leqslant E_1^{s_1} \vee \cdots \vee E_k^{s_k}$. In either case, $E_i^n \leqslant B \vee (0:A)$. Hence $BA:A \leqslant B \vee (0:A)$. Q.E.D.

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