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SOME l-SIMPLE PATHOLOGICAL LATTICE-ORDERED GROUPS

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ABSTRACT. The purpose of this note is to explore two recently discovered lattice-ordered groups, and to establish that they are l-simple, i.e., have no proper l-ideals. These groups are "pathologically o-2-transitive" ordered permutation groups acting on the real line. The two groups exhibit even nastier properties than previous pathological groups, and serve as counterexamples to several conjectures about l-groups.

1. **Introduction.** $\mathbf{R}$ will denote the set of real numbers ordered in the usual way, and $A(\mathbf{R})$ the lattice-ordered group of all order-preserving permutations of $\mathbf{R}$, with $f \leqslant g$ if and only if $xf \leqslant xg$ for all $x \in \mathbf{R}$. Let F be an l-subgroup of $A(\mathbf{R})$, i.e., a subgroup which is simultaneously a sublattice. F is said to be *o-2-transitive* if for all $u < v$ and $x < y$ in $\mathbf{R}$, there exists $f \in F$ such that $uf = x$ and $vf = y$. For $f \in F$, let $support(f) = \{x \in \mathbf{R}\colon xf \neq x\}$. An o-2-transitive l-subgroup F of $A(\mathbf{R})$ having no element of bounded support other than the identity element 1 is called *pathologically o-2-transitive*.

The grandfather of all pathologically o-2-transitive groups, which was exhibited by Charles Holland in [4], was $\{g \in A(\mathbf{R})\colon \exists n = n(g) \in \mathbf{Z}^+$ such that $\forall x \in \mathbf{R}, \forall k \in \mathbf{Z}, (x + kn)g = xg + kn\}$. Here $\mathbf{Z}$ denotes the integers and $\mathbf{Z}^+$ the positive integers. This group will be denoted throughout by G. G was shown in [5] to be l-simple, i.e., to have no proper l-ideals (normal convex l-subgroups). In [7], another pathological group was introduced, namely $\{g \in A(\mathbf{R})\colon \forall x \in \mathbf{R}, \exists n = n(g, x) \in \mathbf{Z}^+$ such that $\forall k \in \mathbf{Z}, (x + kn)g = xg + kn\}$, here denoted by H, and H was shown to be l-simple. In [7] and [6] there were established certain properties of this class of groups which led to the name "pathological". It was found [2] that these groups are not always l-simple. Pathological groups played a crucial role in [3] in the theory of a^*-extensions and †-extensions, where the two groups $\tilde{G}$ and $\tilde{H}$ (defined below) arose quite naturally. $(\tilde{H}, \mathbf{R})$ turned out to be the unique †-closure of the groups $(G, \mathbf{R})$, $(H, \mathbf{R})$, and $(\tilde{G}, \mathbf{R})$.

2. **The l-groups $\tilde{G}$ and $\tilde{H}$.** For $f \in A(\mathbf{R})$ and $x \in \mathbf{R}$, let $\Delta(f, x) = xf - x$. Observe that

$$|\Delta(f, x + kn) - \Delta(f, x)| = |(x + kn)f - (xf + kn)|;$$

this fact will be used heavily.

Let

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$$\tilde{G} = \{ g \in A(\mathbf{R}) \colon \forall \varepsilon > 0, \exists n = n(g, \varepsilon) \in \mathbf{Z}^+ \text{ s.t. } \forall x \in \mathbf{R}, \forall k \in \mathbf{Z},$$
$$|\Delta(g, x + kn) - \Delta(g, x)| < \varepsilon \text{ and } |\Delta(g^{-1}, x + kn) - \Delta(g^{-1}, x)| < \varepsilon \}.$$

Let

$$\tilde{H} = \{ g \in A(\mathbf{R}) \colon \forall \varepsilon > 0, \forall x \in \mathbf{R}, \exists n = n(g, \varepsilon, x) \in \mathbf{Z}^+ \text{ s.t. } \forall k \in \mathbf{Z},$$
$$|\Delta(g, x + kn) - \Delta(g, x)| < \varepsilon \text{ and } |\Delta(g^{-1}, x + kn) - \Delta(g^{-1}, x)| < \varepsilon \}.$$

These groups were mentioned briefly in [3] in connection with †-extensions, but without proof even of the fact they are groups. Here we fill in some details, and establish their l-simplicity. Clearly $G \subseteq \tilde{G} \subseteq \tilde{H}$ and $G \subseteq H \subseteq \tilde{H}$.

THEOREM. *$\tilde{G}$ and $\tilde{H}$ are l-simple lattice-ordered groups.*

PROOF. We begin by showing that $\tilde{H}$ is closed under product. Let $g, h \in \tilde{H}$, and let $x \in \mathbf{R}$ and $\varepsilon > 0$. Pick $t_1 < xg < t_2$ such that $t_2 h - t_1 h < \varepsilon/2$. Let $\varepsilon_1 = t_2 h - t_1 h < \varepsilon/2$ and $\varepsilon_2 = \min\{t_2 - xg, xg - t_1, \varepsilon/2\}$. Let

$$n = n(h, \varepsilon_1, t_1) n(h, \varepsilon_1, t_2) n(g, \varepsilon_2, x),$$

and let $k \in \mathbf{Z}$. Now

$$|(x + kn)g - (xg + kn)| = |\Delta(g, x + kn) - \Delta(g, x)| < \varepsilon_2,$$

so that $t_1 + kn < (x + kn)g < t_2 + kn$, and thus

(i) $$(t_1 + kn)h < (x + kn)gh < (t_2 + kn)h.$$

Also,

(ii) $$t_1 h + kn < xgh + kn < t_2 h + kn,$$

and by the definition of ε_1,

(iii) $$(t_1 + kn)h < t_2 h + kn,$$

and

(iv) $$t_1 h + kn < (t_2 + kn)h.$$

Since $(t_2 h + kn) - (t_1 h + kn) = t_2 h - t_1 h < \varepsilon/2$, and since $|(t_i + kn)h - (t_i h + kn)| < \varepsilon/2$ $(i = 1, 2)$, we have

$$|(t_2 + kn)h - (t_1 h + kn)| < \varepsilon \quad \text{and} \quad |(t_2 h + kn) - (t_1 + kn)h| < \varepsilon.$$

In view of the numbered facts, this yields

$$|\Delta(gh, x + kn) - \Delta(gh, x)| = |(x + kn)gh - (xgh + kn)| < \varepsilon.$$

Since $(gh)^{-1} = h^{-1}g^{-1}$, there exists $m \in \mathbf{Z}^+$ such that for all $k \in \mathbf{Z}$, $|\Delta((gh)^{-1}, x + km) - \Delta((gh)^{-1}, x)| < \varepsilon$, and then nm works for both gh and $(gh)^{-1}$. Therefore $gh \in \tilde{H}$.

In the above argument, if $g, h \in \tilde{G}$, then n and m can be chosen independently of x, so that $gh \in \tilde{G}$, *provided h is uniformly continuous*, which we now show. Let $\varepsilon > 0$ and $n = n(h, \varepsilon/3)$. There exists $\delta > 0$ such that if $x, y \in [0, 2n]$ and $|x - y| < \delta$, then $|xh - yh| < \varepsilon/3$. But $|\Delta(h, x + kn) - \Delta(h, x)| < \varepsilon/3$ for all $k \in \mathbf{Z}$. A simple computation shows that if $u, v \in \mathbf{R}$ and $|u - v| < \delta$, then $|uh - vh| < \varepsilon$. Hence h is uniformly continuous.

We have shown that $\tilde{H}$ and $\tilde{G}$ are subgroups of $A(\mathbf{R})$, and clearly they are l-subgroups. Since $G \subseteq \tilde{G} \subseteq \tilde{H}$ and G is o-2-transitive, so are $\tilde{H}$ and $\tilde{G}$. Moreover, $\tilde{H}$ (and thus also $\tilde{G}$) are pathological. For suppose $1 \neq h \in \tilde{H}$, so that there exists $x \in \mathbf{R}$ such that $xh \neq x$. Let $\varepsilon = |\Delta(h, x)|/2 > 0$. Then there exists $n = n(h, \varepsilon, x) \in \mathbf{Z}^+$ such that $|\Delta(h, x + kn) - \Delta(h, x)| < \varepsilon$ for all $k \in \mathbf{Z}$. Thus $(x + kn)h \neq x + kn$ for all $k \in \mathbf{Z}$.

Recall that G is l-simple [4]. Thus to prove that $\tilde{H}$ is l-simple, it is enough to show that if $M \neq \{1\}$ is an l-ideal of $\tilde{H}$, then some element of M exceeds a (strictly) positive element of G, since then M will contain all translations. (Let $1 < f \in \tilde{H}$ and $n = n(f, 1, 0)$. Then $xf - x < 0f + n + 1$ for all $x \in \mathbf{R}$, so f is bounded by a translation.) Let $1 < h \in M$. There exists $x \in \mathbf{R}$ such that $x < xh$. Let $\varepsilon = (xh - x)/2 > 0$ and $n = n(h, \varepsilon, x)$. Then $(x + kn)h > xh + kn - \varepsilon$ for all $k \in \mathbf{Z}$. Hence $(x + kn)h > x + kn + \varepsilon$. Thus for all $s \in [x + kn, x + kn + \varepsilon/2]$, we have

$$sh - s > (x + kn)h - (x + kn + \varepsilon/2) > \varepsilon/2.$$

Then h exceeds a positive $g \in G$ which moves points near each $x + kn$ and agrees with the identity elsewhere. As indicated above, it follows that $\tilde{H}$ is l-simple.

This argument also shows that $\tilde{G}$ is l-simple. Alternatively, the l-simplicity of $\tilde{G}$ can be deduced from a routine modification of the argument in [5] for that of G.

G is actually simple as a group [8], and so is H. It is not known whether $\tilde{G}$ and $\tilde{H}$ are simple as groups.

PROPOSITION. *$(\tilde{H}, \mathbf{R})$ and $(\tilde{G}, \mathbf{R})$ are not isomorphic as ordered permutation groups.*

PROOF. It suffices to show that $\tilde{H} \neq \tilde{G}$. For it was shown in [3] (without determining whether $\tilde{H} = \tilde{G}$) that $(\tilde{H}, \mathbf{R})$ is the unique $+$-closure of $(\tilde{G}, \mathbf{R})$, so that $(\tilde{H}, \mathbf{R})$ is $+$-closed, while if $\tilde{H} \neq \tilde{G}$, $(\tilde{G}, \mathbf{R})$ is not $+$-closed. (See [3] for definitions.)

To construct $h \in \tilde{H} \setminus \tilde{G}$, we proceed as follows: Let $S_1 = \{k \in \mathbf{Z}: k \equiv 0 \pmod 2\}$. For $i > 1$, let s_i be that integer not in $S_1 \cup \cdots \cup S_{i-1}$ having smallest absolute value (taking the positive choice in case of ties), and let S_i be $\{k \in \mathbf{Z}: k \equiv s_i \pmod{2^i}\}$. The S_i's partition $\mathbf{Z}$. Let g_i be the order-preserving permutation of $[0, 1]$ defined by $xg_i = x$ for $x \in [0, 1/(i + 2)]$ and $xg_i = (x + 1)/2$ for $x \in [1/(i + 1), 1]$, with the remainder filled in linearly. For $y_i = ((1/i + 1) + (1/i + 2))/2$, $|y_i g_i - y_i g_j| > 1/8$ for all $i \neq j \in \mathbf{Z}^+$. Now define h on $[k, k + 1]$, $k \in \mathbf{Z}$, by the rule $xh = (x - k)g_i + k$, where $k \in S_i$.

Then $h \in H \subseteq \tilde{H}$ since when $x \in [k, k + 1]$ and $k \in S_i$, $n(h, x) = 2^i$. Moreover, $h \notin \tilde{G}$, for when $k \in S_i$, 2^i is the smallest value for $n(h, 1/8, x = k + y_i)$. Hence $\tilde{H} \neq \tilde{G}$, proving the proposition.

Presumably $\tilde{H}$ and $\tilde{G}$ are not isomorphic as l-groups, but we have not been able to prove this. G is not isomorphic even as a group to $\tilde{G}$, $\tilde{H}$ or H since there exists $1 \neq z \in G$ such that each $g \in G$ commutes with some positive power of z, whereas it can be shown that this is false for the other three groups. Also, H is not isomorphic as an l-group to either $\tilde{G}$ or $\tilde{H}$. This can be

seen in terms of order units. If F is an l-group, $u \in F^+$ is called a *strong order unit* if every $f \in F$ is exceeded by some power of u, and we shall call u a *mild order unit* if every $f \in F$ fails to exceed some power of u. In all four of our pathological groups, positive translations are strong order units. In H, every mild order unit is a strong order unit, the two conditions being satisfied by those u's in H^+ which have no fixed points. On the other hand, in the next section we shall construct $g \in \tilde{G}^+ \subseteq \tilde{H}^+$ fixing precisely one point. Then g is a mild order unit, but not a strong order unit.

3. **Pathological groups as counterexamples**. In [4, Theorem 7 and p. 432], Charles Holland proved a theorem and stated a conjecture about transitive ordered permutation groups acting on arbitrary totally ordered sets S. Here we consider the special case in which the group acts on $\mathbf{R}$.

A *prime* subgroup of an l-group F is a convex l-subgroup P such that $f \wedge g = 1$ implies $f \in P$ or $g \in P$. A prime subgroup containing no l-ideal of F other than $\{1\}$ is called a *representing* subgroup. For any transitive lattice-ordered permutation group (F, S), the point stabilizers $F_s = \{f \in F: sf = s\}$ are representing subgroups of F. The condition that these stabilizers be *maximal* representing subgroups of F is of interest because it is equivalent to a certain kind of "minimality" of S.

THEOREM (HOLLAND). *Let $(F, \mathbf{R})$ be a transitive lattice-ordered permutation group having a nonidentity element of bounded support, and let θ be an l-isomorphism of F onto some transitive l-subgroup of some $A(S)$ such that the point stabilizers of $(F\theta, S)$ are maximal representing subgroups of $F\theta$. Then there exists an o-isomorphism ψ from $\mathbf{R}$ onto S which induces θ in the sense that for all $f \in F$, $f\theta = \psi^{-1}f\psi$.*

Holland says that "it seems likely that the theorem remains true" without the hypothesis that F contains a nonidentity element of bounded support. However, such is not the case. The conclusion of the theorem requires that every maximal representing subgroup of F be a point stabilizer F_r for some $r \in \mathbf{R}$ (see [4]). We now construct a maximal representing subgroup K of the original pathological group G which is not a stabilizer of $(G, \mathbf{R})$. (The construction works equally well for the groups H, $\tilde{G}$, and $\tilde{H}$.)

Let $T_i = b_i + 4^iZ$, $i = 1, 2, \ldots,$ where $b_1 = 1$ and for $i > 1$, b_i is the second smallest positive integer in T_{i-1}. Then $T_1 \supset T_2 \supset \ldots$ forms a descending tower having empty intersection. Enlarge this tower to an ultrafilter $\mathfrak{F}$ on $\mathbf{R}$. Let $P = \{g \in G | \exists F_g \in \mathfrak{F}$ such that $rg = r \forall r \in F_g\}$, a prime subgroup of G. Given any $r \in \mathbf{R}$, we can pick a T_i that does not contain r and construct an order-preserving permutation g which moves r, has period 4^i (so that $g \in G$), and fixes each $t \in T_i$ (so that $g \in P$). Hence P is not contained in any stabilizer G_r. $P \neq G$ since each $g \in P$ fixes some $r \in \mathbf{R}$. Since G has a strong order unit, Zorn's lemma guarantees that P is contained in a maximal proper prime subgroup K of G. Since K is proper and G is l-simple, K is a (maximal) representing subgroup of G. K is not equal to any G_r, for then $P \subseteq G_r$. Thus the key hypothesis of Holland's theorem is in fact needed. We remark without proof that the right regular representation of G on the totally ordered set G/K of right cosets of K is also pathologically o-2-transitive.

Now we show that $\tilde{G}$ and $\tilde{H}$ enjoy a very unpleasant property making them even more perverse than the earlier pathological groups: They contain an element g which fixes precisely one point. This answers a question of existence posed by McCleary and resolves in the negative a conjecture of Richard Ball (see [1]).

First we construct $\bar{g} \in A(\mathbf{R})$ by setting

$$x\bar{g} = \begin{cases} x \text{ if } x \text{ is an integer divisible by 4,} \\ x + \frac{1}{2} \text{ if } x \in \left[2p - \frac{1}{4}, 2p + \frac{1}{4}\right], \quad p \text{ an odd integer,} \end{cases}$$

and then filling in the remainder linearly. Now, for $n \geqslant 2$ and q odd, we set $xg = x + 1/2^n$ throughout the interval containing $2^n q$ and extending in each direction until the graph of $x + 1/2^n$ first crosses the graph of $\bar{g}$, and we set $xg = x\bar{g}$ for all other x. It is easily seen that for $2^m < \varepsilon$, we may take $n(g, \varepsilon) = 2^m$, so that $g \in \tilde{G} \subseteq \tilde{H}$. By construction, g fixes only 0.

Let K be an l-group. A convex l-subgroup C is said to be *closed* if whenever a subset of C has a supremum in K, that supremum lies in C. If $k \in K$, $K(k)^*$ denotes the closed convex l-subgroup generated by k. Now let (K, S) be a lattice-ordered permutation group, and let $\bar{S}$ be the completion of S by Dedekind cuts (without endpoints). If $T \subseteq \bar{S}$, let $K_T = \{k \in K\colon tk = t \ \forall t \in T\}$. If $D \subseteq G$, let $\bar{F}xD = \{s \in \bar{S}\colon sd = s \ \forall d \in D\}$. For any $k \in K$, we clearly have

$$\bar{F}x(k) \cap \left\{s \in \bar{S}\colon G_s \text{ is closed}\right\} \subseteq \bar{F}x(K(k)^*) \subseteq \bar{F}x(k).$$

In [1], Ball introduced the concept of a *full* convex l-subgroup of an l-group K, meaning one which is a union of closed convex l-subgroups. When (K, S) is a permutation group, F denotes $\{s \in \bar{S}\colon G_s \text{ is full}\}$. Clearly we can improve the above result to say $\bar{F}x(k) \cap F \subseteq \bar{F}x(K(k)^*) \subseteq \bar{F}x(k)$. Ball conjectured [1, 5.1.10] that for transitive groups, $\bar{F}x(k) \cap F = \bar{F}x(K(k)^*)$ for all $k \in K^+$. In view of the fact that $K_{\bar{F}xC} = C$ when C is closed in K, this would imply that $K(k)^* = K_{\bar{F}x(k) \cap F}$ for all $k \in K^+$ [1, 5.1.9].

However, it has been observed in a discussion between Ball and the present authors that $\tilde{G}$ and $\tilde{H}$ provide counterexamples to these conjectures. As Ball observed in [1], $(G, \mathbf{R})$ has full stabilizers, for $G_0 = \bigcup\{G_{n\mathbf{Z}}\colon n \in \mathbf{Z}^+\}$, a union of closed subgroups. The same is true of $(H, \mathbf{R})$. But for $\tilde{G}$ and $\tilde{H}$, the stabilizers are not full. Let $g \in \tilde{G}$ fix only 0, as above. Let z denote the translation $x \mapsto x + 1$, and let $r \in \mathbf{R} \setminus [-1, 0]$. Since g fixes only 0, there exists a power g^{n_r} of g such that $rg^{n_r} > r + 1$. Then $z = \sup\{g^{n_r} \wedge z\}$ since if $g^{n_r} \wedge z \leqslant f < z$ for all r, zf^{-1} would have bounded support. Thus, letting K be either $\tilde{G}$ or $\tilde{H}$, we have $z \in K(g)^*$, so that $K(g)^* = K$. Since $g \in K_0$, K_0 cannot be full. Hence F is empty, so that Ball's weaker conjecture would here say that for any $k \in K^+$, $K(k)^* = K$. This fails for any k lying in a proper closed convex l-subgroup of K, e.g., in $K_{\mathbf{Z}}$.

4. **Other pathological groups**. Our four examples of pathological groups all have strong order units. The examples below will lack strong order units, and thus will not be isomorphic as l-groups to any of the above examples. First, the restriction $(G_0|\mathbf{R}^+, \mathbf{R}^+)$ of the stabilizer G_0 is pathological. Those of its elements which are bounded below form a proper l-ideal, so this example is

not l-simple. A smaller proper l-ideal is given by $M = \{g \in G_0|\mathbf{R}^+ : \exists n = n(g) \in \mathbf{Z}^+$ such that $\forall x \in \mathbf{R}^+, (x + kn)g = xg + kn$ for all $k \in \mathbf{Z}^+$ and there exists a neighborhood of 0 throughout which $xg = x\}$. M is l-simple. For let $1 < g, h \in M^+$, and let $n = n(g)n(h)$. A standard compactness argument shows that g is exceeded by a product of conjugates (by elements of period n) of h.

5. **Epilogue**. For convenience, we tabulate the properties of the six pathological groups mentioned above. No two of these are isomorphic as l-groups except, conceivably, $\tilde{G}$ and $\tilde{H}$, and even these are not isomorphic as ordered permutation groups.

	l-simple	strong order unit	element fixing precisely one point	full stabilizers
G	Yes	Yes	No	Yes
H	Yes	Yes	No	Yes
$\tilde{G}$	Yes	Yes	Yes	No
$\tilde{H}$	Yes	Yes	Yes	No
$G_0\|\mathbf{R}^+$	No	No	No	Yes
M	Yes	No	No	Yes

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