SOME $l$-SIMPLE PATHOLOGICAL LATTICE-ORDERED GROUPS

A. M. W. GLASS$^1$ AND STEPHEN H. MCCLEY

Abstract. The purpose of this note is to explore two recently discovered lattice-ordered groups, and to establish that they are $l$-simple, i.e., have no proper $l$-ideals. These groups are “pathologically $o$-2-transitive” ordered permutation groups acting on the real line. The two groups exhibit even nastier properties than previous pathological groups, and serve as counterexamples to several conjectures about $l$-groups.

1. Introduction. $R$ will denote the set of real numbers ordered in the usual way, and $A(R)$ the lattice-ordered group of all order-preserving permutations of $R$, with $f < g$ if and only if $xf < xg$ for all $x \in R$. Let $F$ be an $l$-subgroup of $A(R)$, i.e., a subgroup which is simultaneously a sublattice. $F$ is said to be $o$-2-transitive if for all $u < v$ and $x < y$ in $R$, there exists $f \in F$ such that $uf = x$ and $vf = y$. For $f \in F$, let $support(f) = \{x \in R: xf \neq x\}$. An $o$-2-transitive $l$-subgroup $F$ of $A(R)$ having no element of bounded support other than the identity element 1 is called pathologically $o$-2-transitive.

The grandfather of all pathologically $o$-2-transitive groups, which was exhibited by Charles Holland in [4], was $\{g \in A(R): \exists n = n(g) \in \mathbb{Z}^+ \text{ such that } \forall x \in R, \forall k \in \mathbb{Z}, (x + kn)g = xg + kn\}$. Here $\mathbb{Z}$ denotes the integers and $\mathbb{Z}^+$ the positive integers. This group will be denoted throughout by $G$. $G$ was shown in [5] to be $l$-simple, i.e., to have no proper $l$-ideals (normal convex $l$-subgroups). In [7], another pathological group was introduced, namely $\{g \in A(R): \forall x \in R, \exists n = n(g, x) \in \mathbb{Z}^+ \text{ such that } \forall k \in \mathbb{Z}, (x + kn)g = xg + kn\}$, here denoted by $H$, and $H$ was shown to be $l$-simple. In [7] and [6] there were established certain properties of this class of groups which led to the name “pathological”. It was found [2] that these groups are not always $l$-simple. Pathological groups played a crucial role in [3] in the theory of $a^*$-extensions and $\dagger$-extensions, where the two groups $\tilde{G}$ and $\tilde{H}$ (defined below) arose quite naturally. $(\tilde{H}, R)$ turned out to be the unique $\dagger$-closure of the groups $(G, R)$, $(H, R)$, and $(\tilde{G}, R)$.

2. The $l$-groups $\tilde{G}$ and $\tilde{H}$. For $f \in A(R)$ and $x \in R$, let $\Delta(f, x) = xf - x$. Observe that 

$$|\Delta(f, x + kn) - \Delta(f, x)| = |(x + kn)f - (xf + kn)|;$$

this fact will be used heavily.

Let

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\[ \tilde{G} = \{ g \in A(R) : \forall \varepsilon > 0, \exists n = n(g, \varepsilon) \in Z^+ \text{ s.t. } \forall x \in R, \forall k \in Z, \]
\[ |\Delta(g, x + kn) - \Delta(g, x)| < \varepsilon \text{ and } |\Delta(g^{-1}, x + kn) - \Delta(g^{-1}, x)| < \varepsilon \}. \]

Let
\[ \tilde{H} = \{ g \in A(R) : \forall \varepsilon > 0, \forall x \in R, \exists n = n(g, \varepsilon, x) \in Z^+ \text{ s.t. } \forall x \in Z, \]
\[ |\Delta(g, x + kn) - \Delta(g, x)| < \varepsilon \text{ and } |\Delta(g^{-1}, x + kn) - \Delta(g^{-1}, x)| < \varepsilon \}. \]

These groups were mentioned briefly in [3] in connection with $\dagger$-extensions, but without proof even of the fact they are groups. Here we fill in some details, and establish their $l$-simplicity. Clearly $G \subseteq \tilde{G} \subseteq \tilde{H}$ and $G \subseteq H \subseteq \tilde{H}$.

**Theorem.** $\tilde{G}$ and $\tilde{H}$ are $l$-simple lattice-ordered groups.

**Proof.** We begin by showing that $\tilde{H}$ is closed under product. Let $g, h \in \tilde{H}$, and let $x \in R$ and $\varepsilon > 0$. Pick $t_1 < x_g < t_2$ such that $t_2h - t_1h < \varepsilon/2$. Let $\varepsilon_1 = t_2h - t_1h < \varepsilon/2$ and $\varepsilon_2 = \min\{t_2 - x_g, x_g - t_1, \varepsilon/2\}$. Let
\[ n = n(h, \varepsilon_1, t_1)n(h, \varepsilon_1, t_2)n(g, \varepsilon_2, x), \]
and let $k \in Z$. Now
\[ |(x + kn)g - (xg + kn)| = |\Delta(g, x + kn) - \Delta(g, x)| < \varepsilon_2, \]
so that $t_1 + kn < (x + kn)g < t_2 + kn$, and thus
(i) \[(t_1 + kn)h < (x + kn)gh < (t_2 + kn)h.\]
Also,
(ii) \[t_1h + kn < xgh + kn < t_2h + kn,\]
and by the definition of $\varepsilon_1$,
(iii) \[(t_1 + kn)h < t_2h + kn,\]
and
(iv) \[t_1h + kn < (t_2 + kn)h.\]
Since $(t_2h + kn) - (t_1h + kn) = t_2h - t_1h < \varepsilon/2$, and since $|(t_i + kn)h - (t_ih + kn)| < \varepsilon/2$ $(i = 1, 2)$, we have
\[ |(t_2 + kn)h - (t_1h + kn)| < \varepsilon \text{ and } |(t_2h + kn) - (t_1 + kn)h| < \varepsilon. \]
In view of the numbered facts, this yields
\[ |\Delta(gh, x + kn) - \Delta(gh, x)| = |x + kn)|gh - (xgh + kn)| < \varepsilon. \]
Since $(gh)^{-1} = h^{-1}g^{-1}$, there exists $m \in Z^+$ such that for all $k \in Z$, $|\Delta((gh)^{-1}, x + km) - \Delta((gh)^{-1}, x)| < \varepsilon$, and then $nm$ works for both $gh$ and $(gh)^{-1}$. Therefore $gh \in \tilde{H}$.

In the above argument, if $g, h \in \tilde{G}$, then $n$ and $m$ can be chosen independently of $x$, so that $gh \in \tilde{G}$, provided $h$ is uniformly continuous, which we now show. Let $\varepsilon > 0$ and $n = n(h, \varepsilon/3)$. There exists $\delta > 0$ such that if $x, y \in [0, 2\pi]$ and $|x - y| < \delta$, then $|xh - yh| < \varepsilon/3$. But $|\Delta(h, x + kn) - \Delta(h, x)| < \varepsilon/3$ for all $k \in Z$. A simple computation shows that if $u, v \in R$ and $|u - v| < \delta$, then $|uh - vh| < \varepsilon$. Hence $h$ is uniformly continuous.
We have shown that \( \tilde{H} \) and \( \tilde{G} \) are subgroups of \( A(\mathbb{R}) \), and clearly they are \( l \)-subgroups. Since \( G \subseteq \tilde{G} \subseteq \tilde{H} \) and \( G \) is \( o \)-2-transitive, so are \( \tilde{H} \) and \( \tilde{G} \). Moreover, \( \tilde{H} \) (and thus also \( \tilde{G} \)) are pathological. For suppose \( 1 \neq h \in \tilde{H} \), so that there exists \( x \in \mathbb{R} \) such that \( xh \neq x \). Let \( \epsilon = \frac{|\Delta(h, x)|}{2} > 0 \). Then there exists \( n = n(h, \epsilon, x) \in \mathbb{Z}^+ \) such that \( |\Delta(h, x + kn) - \Delta(h, x)| < \epsilon \) for all \( k \in \mathbb{Z} \). Thus \( (x + kn)h \neq x + kn \) for all \( k \in \mathbb{Z} \).

Recall that \( G \) is \( l \)-simple [4]. Thus to prove that \( \tilde{H} \) is \( l \)-simple, it is enough to show that if \( M \neq \{1\} \) is an \( l \)-ideal of \( H \), then some element of \( M \) exceeds a (strictly) positive element of \( G \), since then \( M \) will contain all translations. (Let \( 1 < f \in \tilde{H} \) and \( n = n(f, 1, 0) \). Then \( xf - x < 0f + n + 1 \) for all \( x \in \mathbb{R} \), so \( f \) is bounded by a translation.) Let \( 1 < h \in M \). There exists \( x \in \mathbb{R} \) such that \( x < xh \). Let \( \epsilon = \frac{(xh - x)}{2} > 0 \) and \( n = n(h, \epsilon, x) \). Then \( (x + kn)h > xh + kn - \epsilon \) for all \( k \in \mathbb{Z} \). Hence \( (x + kn)h > x + kn + \epsilon \). Thus for all \( s \in [x + kn, x + kn + \epsilon/2] \), we have

\[
sh - s > (x + kn)h - (x + kn + \epsilon/2) > \epsilon/2.
\]

Then \( h \) exceeds a positive \( g \in G \) which moves points near each \( x + kn \) and agrees with the identity elsewhere. As indicated above, it follows that \( \tilde{H} \) is \( l \)-simple.

This argument also shows that \( \tilde{G} \) is \( l \)-simple. Alternatively, the \( l \)-simplicity of \( \tilde{G} \) can be deduced from a routine modification of the argument in [5] for that of \( G \).

\( G \) is actually simple as a group [8], and so is \( H \). It is not known whether \( \tilde{G} \) and \( \tilde{H} \) are simple as groups.

**Proposition.** \((\tilde{H}, R) \) and \((\tilde{G}, R) \) are not isomorphic as ordered permutation groups.

**Proof.** It suffices to show that \( \tilde{H} \neq \tilde{G} \). For it was shown in [3] (without determining whether \( \tilde{H} = \tilde{G} \)) that \((\tilde{H}, R)\) is the unique \(+\)-closure of \((\tilde{G}, R)\), so that \((\tilde{H}, R)\) is \(+\)-closed, while if \( \tilde{H} \neq \tilde{G} \), \((\tilde{G}, R)\) is not \(+\)-closed. (See [3] for definitions.)

To construct \( h \in \tilde{H} \setminus \tilde{G} \), we proceed as follows: Let \( S_i = \{ k \in \mathbb{Z}; k \equiv 0 \pmod{2} \} \). For \( i > 1 \), let \( s_i \) be that integer not in \( S_1 \cup \cdots \cup S_{i-1} \) having smallest absolute value (taking the positive choice in case of ties), and let \( S_i \) be \( \{ k \in \mathbb{Z}; k \equiv s_i \pmod{2i} \} \). The \( S_i \)'s partition \( \mathbb{Z} \). Let \( g_i \) be the order-preserving permutation of \([0, 1]\) defined by \( xg_i = x \) for \( x \in [0, 1/(i + 2)] \) and \( xg_i = (x + 1)/2 \) for \( x \in [1/(i + 1), 1] \), with the remainder filled in linearly. For \( y_i = ((1/i + 1) + (1/i + 2))/2 \), \( |y_ig_i - y_ig_j| > 1/8 \) for all \( i \neq j \in \mathbb{Z}^+ \). Now define \( h \) on \([k, k + 1], k \in \mathbb{Z} \), by the rule \( xh = (x - k)g_i + k \), where \( k \in S_i \).

Then \( h \in H \subseteq \tilde{H} \) since when \( x \in [k, k + 1] \) and \( k \in S_i \), \( n(h, x) = 2^i \). Moreover, \( h \not\in \tilde{G} \), for when \( k \in S_i \), \( 2^i \) is the smallest value for \( n(h, 1/8, x = k + y_i) \). Hence \( \tilde{H} \neq \tilde{G} \), proving the proposition.

Presumably \( \tilde{H} \) and \( \tilde{G} \) are not isomorphic as \( l \)-groups, but we have not been able to prove this. \( G \) is not isomorphic even as a group to \( \tilde{G} \) or \( H \) since there exist \( 1 \neq z \in G \) such that each \( g \in G \) commutes with some positive power of \( z \), whereas it can be shown that this is false for the other three groups. Also, \( H \) is not isomorphic as an \( l \)-group to either \( \tilde{G} \) or \( \tilde{H} \). This can be
seen in terms of order units. If \( F \) is an \( I \)-group, \( u \in F^+ \) is called a **strong order unit** if every \( f \in F \) is exceeded by some power of \( u \), and we shall call \( u \) a **mild order unit** if every \( f \in F \) fails to exceed some power of \( u \). In all four of our pathological groups, positive translations are strong order units. In \( H \), every mild order unit is a strong order unit, the two conditions being satisfied by those \( u \)'s in \( H^+ \) which have no fixed points. On the other hand, in the next section we shall construct \( g \in \tilde{G}^+ \subseteq \tilde{H}^+ \) fixing precisely one point. Then \( g \) is a mild order unit, but not a strong order unit.

3. Pathological groups as counterexamples. In [4, Theorem 7 and p. 432], Charles Holland proved a theorem and stated a conjecture about transitive ordered permutation groups acting on arbitrary totally ordered sets \( S \). Here we consider the special case in which the group acts on \( \mathbb{R} \).

A **prime** subgroup of an \( I \)-group \( F \) is a convex \( I \)-subgroup \( P \) such that \( f A g \neq 1 \) implies \( f \in P \) or \( g \in P \). A prime subgroup containing no \( I \)-ideal of \( F \) other than \( \{1\} \) is called a **representing** subgroup. For any transitive lattice-ordered permutation group \((F, S)\), the point stabilizers \( F_r = \{f \in F : sf = s\} \) are representing subgroups of \( F \). The condition that these stabilizers be **maximal** representing subgroups of \( F \) is of interest because it is equivalent to a certain kind of "minimality" of \( S \).

**Theorem (Holland).** Let \((F, \mathbb{R})\) be a transitive lattice-ordered permutation group having a nonidentity element of bounded support, and let \( \theta \) be an \( I \)-isomorphism of \( F \) onto some transitive \( I \)-subgroup of some \( A(S) \) such that the point stabilizers of \((F \theta, S)\) are maximal representing subgroups of \( F \theta \). Then there exists an \( o \)-isomorphism \( \psi \) from \( \mathbb{R} \) onto \( S \) which induces \( \theta \) in the sense that for all \( f \in F \), \( f \theta \psi = \psi^{-1}f\psi \).

Holland says that "it seems likely that the theorem remains true" without the hypothesis that \( F \) contains a nonidentity element of bounded support. However, such is not the case. The conclusion of the theorem requires that every maximal representing subgroup of \( F \) be a point stabilizer \( F_r \) for some \( r \in \mathbb{R} \) (see [4]). We now construct a maximal representing subgroup \( K \) of the original pathological group \( G \) which is not a stabilizer of \((G, \mathbb{R})\). (The construction works equally well for the groups \( H \), \( \tilde{G} \), and \( \tilde{H} \).)

Let \( T_i = b_i + 4^i \mathbb{Z}, \) \( i = 1, 2, \ldots \), where \( b_1 = 1 \) and for \( i > 1 \), \( b_i \) is the second smallest positive integer in \( T_{i-1} \). Then \( T_1 \supset T_2 \supset \ldots \) forms a descending tower having empty intersection. Enlarge this tower to an ultrafilter \( \mathcal{G} \) on \( \mathbb{R} \). Let \( P = \{g \in G | \exists F_g \in \mathcal{G} \text{ such that } rg = r\forall r \in F_g\} \), a prime subgroup of \( G \). Given any \( r \in \mathbb{R} \), we can pick a \( T_i \) that does not contain \( r \) and construct an order-preserving permutation \( g \) which moves \( r \), has period \( 4^i \) (so that \( g \in G \)), and fixes each \( t \in T_i \) (so that \( g \in P \)). Hence \( P \) is not contained in any stabilizer \( G_r \). \( P \neq G \) since each \( g \in P \) fixes some \( r \in \mathbb{R} \). Since \( G \) has a strong order unit, Zorn's lemma guarantees that \( P \) is contained in a maximal proper prime subgroup \( K \) of \( G \). Since \( K \) is proper and \( G \) is \( I \)-simple, \( K \) is a (maximal) representing subgroup of \( G \). \( K \) is not equal to any \( G_r \), for then \( P \subseteq G_r \). Thus the key hypothesis of Holland's theorem is in fact needed. We remark without proof that the right regular representation of \( G \) on the totally ordered set \( G/K \) of right cosets of \( K \) is also pathologically \( o \)-2-transitive.
Now we show that \( \tilde{G} \) and \( \tilde{H} \) enjoy a very unpleasant property making them even more perverse than the earlier pathological groups: They contain an element \( g \) which fixes precisely one point. This answers a question of existence posed by McCleary and resolves in the negative a conjecture of Richard Ball (see [1]).

First we construct \( \tilde{g} \in A(\mathbb{R}) \) by setting

\[
x\tilde{g} = \begin{cases} x & \text{if } x \text{ is an integer divisible by } 4, \\ x + \frac{1}{2} & \text{if } x \in [2p - \frac{1}{4}, 2p + \frac{1}{4}] , \quad p \text{ an odd integer}, \end{cases}
\]

and then filling in the remainder linearly. Now, for \( n \geq 2 \) and \( q \) odd, we set

\[
xg = x + 1/2^n \text{ throughout the interval containing } 2^nq \text{ and extending in each direction until the graph of } x + 1/2^n \text{ first crosses the graph of } \tilde{g},
\]

and we set \( xg = x\tilde{g} \) for all other \( x \). It is easily seen that for \( 2^m < e \), we may take \( n(g, e) = 2^m \), so that \( g \in \tilde{G} \subseteq \tilde{H} \). By construction, \( g \) fixes only 0.

Let \( K \) be an \( \ell \)-group. A convex \( \ell \)-subgroup \( C \) is said to be closed if whenever a subset of \( C \) has a supremum in \( K \), that supremum lies in \( C \). If \( k \in K \), \( K(k)^* \) denotes the closed convex \( \ell \)-subgroup generated by \( k \). Now let \( (K, S) \) be a lattice-ordered permutation group, and let \( \tilde{S} \) be the completion of \( S \) by Dedekind cuts (without endpoints). If \( T \subseteq \tilde{S} \), let \( K_T = \{ k \in K : tk = t \forall t \in T \} \). If \( D \subseteq G \), let \( F_xD = \{ s \in \tilde{S} : xs = s \forall d \in D \} \). For any \( k \in K \), we clearly have

\[
F_x(k) \cap \{ s \in \tilde{S} : G_s \text{ is closed} \} \subseteq F_x(K(k)^*) \subseteq F_x(k).
\]

In [1], Ball introduced the concept of a full convex \( \ell \)-subgroup of an \( \ell \)-group \( K \), meaning one which is a union of closed convex \( \ell \)-subgroups. When \( (K, S) \) is a permutation group, \( F \) denotes \( \{ s \in \tilde{S} : G_s \text{ is full} \} \). Clearly we can improve the above result to say \( F_x(k) \cap F \subseteq F_x(K(k)^*) \subseteq F_x(k) \). Ball conjectured [1, 5.1.10] that for transitive groups, \( F_x(k) \cap F = F_x(K(k)^*) \) for all \( k \in K^+ \). In view of the fact that \( K_F = C \) when \( C \) is closed in \( K \), this would imply that \( K(k)^* = F_x(k) \cap F \) for all \( k \in K^+ \) [1, 5.1.9].

However, it has been observed in a discussion between Ball and the present authors that \( \tilde{G} \) and \( \tilde{H} \) provide counterexamples to these conjectures. As Ball observed in [1], \( (G, \mathbb{R}) \) has full stabilizers, for \( G_0 = \bigcup \{ G_n\mathbb{Z} : n \in \mathbb{Z}^+ \} \), a union of closed subgroups. The same is true of \( (H, \mathbb{R}) \). But for \( \tilde{G} \) and \( \tilde{H} \), the stabilizers are not full. Let \( g \in \tilde{G} \) fix only 0, as above. Let \( z \) denote the translation \( x \mapsto x + 1 \), and let \( r \in \mathbb{R} \setminus [-1, 0] \). Since \( g \) fixes only 0, there exists a power \( g^n \) of \( g \) such that \( rg^n > r + 1 \). Then \( z = \sup \{ g^n \wedge z \} \) since if \( g^n \wedge z < f < z \) for all \( r, zf^{-1} \) would have bounded support. Thus, letting \( K \) be either \( \tilde{G} \) or \( \tilde{H} \), we have \( z \in K(g)^* \), so that \( K(g)^* = K \). Since \( g \in K_0 \), \( K_0 \) cannot be full. Hence \( F \) is empty, so that Ball’s weaker conjecture would here say that for any \( k \in K^+ \), \( K(k)^* = K \). This fails for any \( k \) lying in a proper closed convex \( \ell \)-subgroup of \( K \), e.g., in \( K_Z \).

4. Other pathological groups. Our four examples of pathological groups all have strong order units. The examples below will lack strong order units, and thus will not be isomorphic as \( \ell \)-groups to any of the above examples. First, the restriction \( (G_0\mathbb{R}^+, \mathbb{R}^+) \) of the stabilizer \( G_0 \) is pathological. Those of its elements which are bounded below form a proper \( \ell \)-ideal, so this example is
not \( l \)-simple. A smaller proper \( l \)-ideal is given by \( M = \{ g \in G_0 | R^+ : \exists n = n(g) \in \mathbb{Z}^+ \text{ such that } \forall x \in R^+, (x + kn)g = xg + kn \text{ for all } k \in \mathbb{Z}^+ \text{ and there exists a neighborhood of } 0 \text{ throughout which } xg = x \}. M \) is \( l \)-simple. For let \( 1 < g, h \in M^+ \), and let \( n = n(g)n(h) \). A standard compactness argument shows that \( g \) is exceeded by a product of conjugates (by elements of period \( n \)) of \( h \).

5. **Epilogue.** For convenience, we tabulate the properties of the six pathological groups mentioned above. No two of these are isomorphic as \( l \)-groups except, conceivably, \( G \) and \( H \), and even these are not isomorphic as ordered permutation groups.

<table>
<thead>
<tr>
<th>Element fixing</th>
<th>( l )-simple</th>
<th>Strong order unit</th>
<th>Precisely one point</th>
<th>Full stabilizers</th>
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<td>Yes</td>
<td>No</td>
<td>Yes</td>
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<td>( H )</td>
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<td>Yes</td>
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<td>No</td>
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