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A CHARACTERIZATION OF MINIMAL HAUSDORFF SPACES

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ABSTRACT. This paper gives a characterization of minimal Hausdorff spaces.

1. **Preliminary definitions and theorems.** A net $\mathcal{O}: \mathfrak{D} \to X$ r-converges[1] to $x_0 \in X$ if for each open V containing x_0, there exists a $d \in \mathfrak{D}$ such that $\mathcal{O}(T_d) \subset \mathrm{cl}\,(V)$ [2]. A net $\mathcal{O}: \mathfrak{D} \to X$ r-accumulates to $x_0 \in X$ if for each open $V \subset X$ containing x_0 and for every $d \in \mathfrak{D}$, $\mathcal{O}(T_d) \cap \mathrm{cl}\,(V) \neq \varnothing$. Theorem 5 of [1] shows that a Hausdorff space X is minimal Hausdorff if and only if each net in X with a unique r-accumulation point is convergent.

A function $f: X \to Y$ has a strongly-closed graph if for each $(x,y) \in G(f)$ ($G(f)$ denotes the graph of f) there exist open sets $U \subset X$ and $V \subset Y$ containing x and y, respectively, such that $(U \times \mathrm{cl}\,(V)) \cap G(f) = \varnothing$ [2]. According to Theorem 7 of [1], each function $f: X \to Y$ of a topological space X into a minimal Hausdorff space Y with strongly-closed graph is continuous. (Note that Example 3 of [1] shows that the strongly-closed graph condition in Theorem 7 of [1] cannot be relaxed to a closed graph condition.)

2. **Main result.** Denote by $\mathfrak{S}$ the class of spaces containing the class of Hausdorff completely normal and fully normal spaces [3].

THEOREM. *A Hausdorff space Y is minimal Hausdorff if and only if for every topological space X belonging to $\mathfrak{S}$, each function $f: X \to Y$ with a strongly-closed graph is continuous.*

PROOF. In view of Theorem 7 of [1], only the sufficiency requires proof. Assume that Y is not minimal Hausdorff. By Theorem 5 of [1] there exists a net $f: \mathfrak{D} \to Y$ with a unique r-accumulation point $q \in Y$ such that f does not converge to q. Let $\infty \notin \mathfrak{D}$ and define $X = \mathfrak{D} \cup \{\infty\}$. Then the power set of $\mathfrak{D}$ together with $\{T_d \cup \{\infty\} | d \in \mathfrak{D}\}$ is a base for a topology σ on X making (X,σ) a fully normal, completely normal Hausdorff space [2], [3]. Define $g: X \to Y$ by $g|\mathfrak{D} = g$ and $g(\infty) = q$. Using the fact that q is the unique r-accumulation point of the net f, it follows that $G(g)$ is strongly-closed. The

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[1] The concepts of r-convergence and r-accumulation point were first introduced by N. V. Veličko under the names of θ-convergence and θ-contact point, respectively, in *H-closed topological spaces*, Mat. Sb. **70 (112)** (1966), 98–112.

identity function $1: \mathfrak{D} \to \mathfrak{D}$ defines a net that converges to ∞. However, since f does not converge to q, there exists an open set $V \subset Y$ containing q with the property that $g(T_d) \cap (Y - V) \neq \varnothing$ for each $d \in \mathfrak{D}$. Consequently, g is not continuous at $x = \infty$. This contradiction establishes the proof.

References

1. Larry L. Herrington and Paul E. Long, *Characterizations of C-compact spaces*, Proc. Amer. Math. Soc. **52** (1975), 417–426.
2. ———, *Characterizations of H-closed spaces*, Proc. Amer. Math. Soc. **48** (1975), 469–475.
3. Shouro Kasahara, *Characterizations of compactness and countable compactness*, Proc. Japan Acad. **49** (1973), 523–524. MR **48** #7207.

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