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QUOTIENTS OF COUNTABLE COMPLETE METRIC SPACES

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The purpose of this note is to prove Theorem 1, and to indicate some generalizations. Theorem 1 provides an alternative approach to the result (see [6, Corollary 1.2]) that the rationals are a quotient image of the irrationals.[1]

THEOREM 1. *Every countable metric space Y is a quotient image of a countable complete metric space X.*

PROOF. For each $y \in Y$, let X_y be Y with $Y - \{y\}$ made discrete (i.e., X_y is the set Y, topologized by calling open all sets $U \cup S$ with U open in Y and $S \subset Y - \{y\}$). Let $X = \sum_{y \in Y} X_y$ (the topological sum of the X_y), and let $f: X \to Y$ be the obvious map.[2] The definitions imply that f is quotient. Clearly X is countable. To show that X is completely metrizable[3], it suffices to show that each X_y has this property. Now X_y is a regular space with a countable base, and is therefore metrizable. Moreover, X_y is the union of the two discrete–hence absolute G_δ–subsets $\{y\}$ and $X_y - \{y\}$, so X_y is also an absolute G_δ and therefore completely metrizable. That completes the proof.

REMARK 1. The conclusion of Theorem 1 remains valid if Y is only assumed to be a countable space with a countable base. (In fact, the proof of [4, Lemma 4.2] implies that such a Y is a continuous open–hence quotient–image of a countable *Hausdorff* space Y' with countable base, and the proof of Theorem 1 goes through to show that such a Y' is the quotient image of a countable complete metric space.)

Our next remark will be applied in [5].

REMARK 2. Theorem 1 can be generalized as follows: Suppose Y is a first-countable, regular space with a countable, closed subset A for which $Y - A$ is completely metrizable. Then Y is a countable-to-one[4] quotient image of a complete metric space.

PROOF. For each $y \in A$, let X_y be Y with $A - \{y\}$ made discrete, and let $X = \sum_{y \in A} X_y$. Then each X_y is metrizable by the Nagata-Smirnov theorem; the rest of the proof is essentially the same as for Theorem 1.

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[1] This result follows from Theorem 1 because every nonempty, complete, separable metric space is a continuous open–hence quotient–image of the irrationals [1, Corollary 4.7].

[2] This construction has also been used, in different contexts, by J. Nagata [7] and E. von Douwen [2].

[3] I.e., metrizable by a complete metric.

[4] Without this requirement, the result is trivial, since *every* first-countable space is a quotient image of the topological sum of its convergent sequences.

Remark 3. The quotient maps in Theorem 1 and Remarks 1 and 2 are automatically biquotient in the sense of [3]. (This follows from [3, Proposition 3.3(c)].)

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