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ON A FIXED POINT THEOREM OF CONTRACTIVE TYPE[1]

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ABSTRACT. A refinement of the proof of a recent fixed point theorem of Caristi is obtained.

The main purpose of this paper is to refine the proof of the following recent result of James Caristi [3].

THEOREM. *Let f be a self-map on a nonempty complete metric space (X, d). Suppose that there exists a lower semicontinuous function V of X into* $[0, \infty)$ *such that*

$$d(x, f(x)) \leqslant V(x) - V(f(x)), \qquad x \in X.$$

Then f has a fixed point.

PROOF. Let $x_0 \in X$ and ν be a nonzero ordinal. Suppose that there exists a net $\{x_\beta\}_{\beta<\nu}$ in X such that:

(i) if β is an isolated nonzero ordinal less than ν, then $x_\beta = f(x_{\beta-1})$;

(ii) if β is a limiting ordinal less than ν, then the net $\{x_\alpha\}_{\alpha<\beta}$ converges to x_β;

(iii) for all $0 \leqslant \alpha < \beta < \nu$, $d(x_\alpha, x_\beta) \leqslant V(x_\alpha) - V(x_\beta)$. If ν is an isolated ordinal, define $x_\nu = f(x_{\nu-1})$ and note that for $\alpha < \nu$,

$$\begin{aligned} d(x_\alpha, x_\nu) &\leqslant d(x_\alpha, x_{\nu-1}) + d(x_{\nu-1}, x_\nu) \\ &\leqslant (V(x_\alpha) - V(x_{\nu-1})) + (V(x_{\nu-1}) - V(x_\nu)) = V(x_\alpha) - V(x_\nu). \end{aligned}$$

Suppose now that ν is a limiting ordinal. We claim that $\{x_\beta\}_{\beta<\nu}$ is a Cauchy net. Suppose not. Then there exists $\varepsilon > 0$ and a strictly increasing sequence $\{\alpha_n\}$ in $(0, \nu)$ such that $d(x_{\alpha_{2n}}, x_{\alpha_{2n-1}}) \geqslant \varepsilon$ for all n. Thus

$$\begin{aligned} \infty &= \sum_{n=1}^{\infty} d(x_{\alpha_{2n}}, x_{\alpha_{2n-1}}) \\ &\leqslant \sum_{n=1}^{\infty} (V(x_{\alpha_{2n}}) - V(x_{\alpha_{2n-1}})) \leqslant V(x_0), \end{aligned}$$

a contradiction. So $\{x_\beta\}_{\beta<\nu}$ is Cauchy and therefore, by completeness of

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(X, d), $\{x_\alpha\}_{\alpha<\nu}$ converges to some point which we shall define to be x_ν. Let $\alpha < \nu$. Since V is lower semicontinuous,

$$d(x_\alpha, x_\nu) = \lim_{\beta<\nu} d(x_\alpha, x_\beta) \leqslant \liminf_{\beta<n}(V(x_\alpha) - V(x_\beta))$$
$$= V(x_\alpha) - \limsup_{\beta<\nu} V(x_\beta) \leqslant V(x_\alpha) - V(x_\nu).$$

This shows that for any ordinal ν, there exists a net $\{x_\beta\}_{\beta<\nu}$ in X with properties (i), (ii), and (iii). If f has no fixed point, then $\{V(x_\alpha)\}_{\alpha<\nu}$ is strictly decreasing and so a contradiction is obtained if we take the cardinality of ν larger than that of X (or $\{1, 2, \ldots\}$). Q.E.D.

The next result follows from the above theorem.

PROPOSITION. *Let (X, d) be a nonempty complete metric space. Let f be a self-map on X. Suppose that there exists a lower semicontinuous function of X into $[0, \infty)$ such that for any x in X with $x \neq f(x)$, there exists y in $X \setminus \{x\}$ such that $d(x, y) \leqslant V(x) - V(y)$. Then f has a fixed point.*

We are informed by the referee and W. A. Kirk that a theorem which is equivalent to Caristi's theorem was announced by I. Ekeland in 1972 [5], a proof [8] of Ekeland's theorem using Zorn's lemma can be found in a recent paper of A. Brønsted [2], and a proof of Caristi's theorem without using Zorn's lemma was obtained by F. E. Browder [1]. Applications of the above theorem of Caristi can be found in [3], [4], [6], [7], [8].

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