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ON THE UNIQUENESS OF CONICAL VECTORS

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ABSTRACT. Certain conical vectors in induced modules for semisimple symmetric Lie algebras have been determined in a previous paper. Here the uniqueness aspects of those results are proved in a simpler way, and a refinement of one of the earlier results is given.

1. **Introduction.** In this paper, we give a new proof of the uniqueness aspects of the main results (Theorems 10.1 and 10.2) of [3], concerning conical vectors and embeddings of induced modules (see Theorem 3.6 and Corollary 3.7). The proof, which we announced in the Introduction of [3], uses the general results on uniqueness of embeddings in [4], as well as an observation of B. Kostant on the usefulness of the action of the center of the enveloping algebra in limiting the possibilities for conical vectors in induced modules (cf. Theorem 3.2 below). We can thus avoid the Kostant-Mostow double transitivity theorem, fundamental commutation relation and transfer principles needed in [3] for proving the uniqueness. (But note that the last two of these were also used in [3] for the existence of conical vectors.)

We also obtain an interesting refinement of Theorem 10.2 of [3] (see Theorem 3.8).

The terminology and setting of [2, §2] and [3, §2] will be used here.

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2. **Consequences of** [4]. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a semisimple symmetric Lie algebra with symmetric decomposition over a field k of characteristic zero, $\mathfrak{a}$ a splitting Cartan subspace of $\mathfrak{p}$, $\mathfrak{m}$ the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, $\Sigma \subset \mathfrak{a}^*$ the system of restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$, $\Sigma_+ \subset \Sigma$ a positive system, $\mathfrak{n} = \coprod \mathfrak{g}^\varphi$ $(\varphi \in \Sigma_+)$ and $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Then $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$. For all $\lambda \in \mathfrak{a}^*$, the linear functional on $\mathfrak{p}$ which vanishes on $\mathfrak{m} \oplus \mathfrak{n}$ and extends λ on $\mathfrak{a}$ defines a one-dimensional $\mathfrak{p}$-module. Let V^λ denote the corresponding induced $\mathfrak{g}$-module. Also, let X^λ be the twisted induced $\mathfrak{g}$-module $V^{\lambda-\rho}$, where $\rho \in \mathfrak{a}^*$ is half the sum of the positive restricted roots with multiplicities counted. (Cf. [3].)

Theorem 1.1 of [4] implies:

THEOREM 2.1. *For all* $\lambda, \mu \in \mathfrak{a}^*$, $\dim \operatorname{Hom}_{\mathfrak{g}}(X^\lambda, X^\mu) \leqslant 1$. *Equivalently, if*

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$S \subset X^\mu$ is the intersection of the conical space (i.e., the space of $\mathfrak{m} \oplus \mathfrak{n}$-invariants) with the weight space for $\mathfrak{a}$ with weight $\lambda - \rho$ (see [4, §2] for the definition), then $\dim S \leqslant 1$. *Moreover, every nonzero $\mathfrak{g}$-module map from X^λ into X^μ is injective. (Cf. [3].)*

3. **Consequences of the action of $\mathfrak{Z}$.** Let $\mathfrak{Z}$ be the center of the universal enveloping algebra $\mathcal{G}$ of $\mathfrak{g}$. It is easy to see that for all $\lambda \in \mathfrak{a}^*$ and $z \in \mathfrak{Z}$, z acts as a scalar on X^λ. Indeed, z must preserve the $\lambda - \rho$-weight space for $\mathfrak{a}$ in X^λ. Thus z must multiply the canonical generator (see [3, §2]) of X^λ by a scalar, and hence z acts as this scalar on all of X^λ. We shall now determine this scalar.

Let $\mathfrak{l}$ be a Cartan subalgebra of $\mathfrak{m}$, so that $\mathfrak{h} = \mathfrak{l} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$ (see [1, Proposition 1.13.7]). Use this decomposition to identify $\mathfrak{h}^*$ with $\mathfrak{l}^* \oplus \mathfrak{a}^*$. By extending the field k if necessary, we may assume that $\mathfrak{l}$ and $\mathfrak{h}$ are splitting Cartan subalgebras of $\mathfrak{m}$ and $\mathfrak{g}$, respectively.

Let $R'_+ \subset \mathfrak{l}^*$ be a system of positive roots of $\mathfrak{m}$ with respect to $\mathfrak{l}$ and let $R''_+ \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ whose restriction to $\mathfrak{a}$ is in Σ_+, so that $\mathfrak{n} = \coprod \mathfrak{g}^\varphi$ $(\varphi \in R''_+)$. Then R_+, defined as $R'_+ \cup R''_+$, is a positive system of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $\mathfrak{n}_\mathfrak{g} = \coprod \mathfrak{g}^\varphi$ $(\varphi \in R_+)$ and $\mathfrak{n}_\mathfrak{m} = \coprod \mathfrak{m}^\varphi$ $(\varphi \in R'_+)$, so that $\mathfrak{n}_\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{n}_\mathfrak{m}$. Also, let $\rho_\mathfrak{g} = \frac{1}{2}\Sigma\varphi$ $(\varphi \in R_+)$ and $\rho_\mathfrak{m} = \frac{1}{2}\Sigma\varphi$ $(\varphi \in R'_+)$. Then $\rho_\mathfrak{g} = \rho + \rho_\mathfrak{m}$. (Indeed, it is sufficient to show that $\rho = \frac{1}{2}\Sigma\varphi$ $(\varphi \in R''_+)$. But both sides have the same restriction to $\mathfrak{a}$. Also, both sides vanish on $\mathfrak{l}$, the right-hand side because R''_+ is stable under the automorphism of $\mathfrak{h}^*$ which is 1 on $\mathfrak{a}^*$ and -1 on $\mathfrak{l}^*$.) Let $W_\mathfrak{g}$ be the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}$, regarded as a group of automorphisms of $\mathfrak{h}^*$.

It is well known that $\mathfrak{Z} \subset \mathcal{H} \oplus \mathcal{G}\mathfrak{n}_\mathfrak{g}$, where $\mathcal{H} \subset \mathcal{G}$ is the universal enveloping algebra of $\mathfrak{h}$. Let $\eta\colon \mathfrak{Z} \to \mathcal{H}$ be the corresponding projection. Identify $\mathcal{H}$ with the algebra of polynomial functions on $\mathfrak{h}^*$, and for all $\lambda \in \mathfrak{h}^*$, let $\gamma_\lambda\colon \mathcal{H} \to k$ be point evaluation at $\lambda - \rho_\mathfrak{g}$. Also let $\chi_\lambda = \gamma_\lambda \circ \eta\colon \mathfrak{Z} \to k$. Since $\rho_\mathfrak{g} = \rho + \rho_\mathfrak{m}$, it is now easy to see that $z \in \mathfrak{Z}$ acts as the scalar $\chi_{\lambda+\rho_\mathfrak{m}}(z)$ on X^λ for all $\lambda \in \mathfrak{a}^*$. But by a theorem of Harish-Chandra, if $\lambda, \mu \in \mathfrak{h}^*$ and $\chi_\lambda = \chi_\mu$, then $\lambda \in W_\mathfrak{g}\mu$ (see for example [1, Proposition 7.4.7]). Hence we have:

PROPOSITION 3.1. *Let $\lambda, \mu \in \mathfrak{a}^*$, and suppose that X^λ and X^μ share a subquotient, i.e., suppose some nonzero subquotient (quotient of submodules) of X^λ is isomorphic to some subquotient of X^μ. Then $\lambda + \rho_\mathfrak{m}$ and $\mu + \rho_\mathfrak{m}$ are conjugate under $W_\mathfrak{g}$.*

Recall that the original field k might have been extended. Assume now that k is again arbitrary of characteristic zero. We can now prove:

THEOREM 3.2. *Let $\varphi \in \Sigma$ and $\lambda, \mu \in \mathfrak{a}^*$ such that $\lambda - \mu$ is of the form $c\varphi$ for some $c \in k$. If X^λ and X^μ share a subquotient, then either $\lambda = \mu$ or $\lambda = s_\varphi\mu$, where s_φ is the Weyl reflection with respect to φ.*

PROOF. We may assume that k is algebraically closed. By Proposition 3.1, $\lambda + \rho_\mathfrak{m}$ and $\mu + \rho_\mathfrak{m}$ are $W_\mathfrak{g}$-conjugate. Let $(\cdot\,,\cdot)$ be the natural nonsingular symmetric bilinear form on $\mathfrak{h}^*$ induced by the Killing form of $\mathfrak{g}$ (cf. [2, §2]). Write $\lambda = a\varphi + \beta$, where $a \in k$, $\beta \in \mathfrak{a}^*$ and $(\varphi, \beta) = 0$. Since $\lambda - \mu = c\varphi$, $\mu = (a - c)\varphi + \beta$. Thus

$$(\lambda + \rho_{\mathfrak{m}}, \lambda + \rho_{\mathfrak{m}}) = a^2(\varphi, \varphi) + (\beta, \beta) + (\rho_{\mathfrak{m}}, \rho_{\mathfrak{m}})$$

and

$$(\mu + \rho_{\mathfrak{m}}, \mu + \rho_{\mathfrak{m}}) = (a - c)^2(\varphi, \varphi) + (\beta, \beta) + (\rho_{\mathfrak{m}}, \rho_{\mathfrak{m}}),$$

since $(\mathfrak{a}^*, \mathfrak{l}^*) = 0$. But these two quantities are equal in view of the $W_{\mathfrak{g}}$-conjugacy of $\lambda + \rho_{\mathfrak{m}}$ and $\mu + \rho_{\mathfrak{m}}$, and so $a^2 = (a - c)^2$. (Recall that $(\varphi, \varphi) \neq 0$ by [2, Lemma 2.2].) Hence either $c = 0$ or $c = 2a$. In the first case, $\lambda = \mu$, and in the second case, $\lambda = s_\varphi \mu$. Q.E.D.

For all $\varphi \in \Sigma$, define $x_\varphi \in \mathfrak{a}$ by the condition $B(x, x_\varphi) = \varphi(x)$ for all $x \in \mathfrak{a}$, where B is the Killing form of $\mathfrak{g}$. Let $h_\varphi = 2x_\varphi/(\varphi, \varphi)$ (using the notation of the last proof), so that $\varphi(h_\varphi) = 2$.

PROPOSITION 3.3. *Let $\varphi \in \Sigma_+$ such that $\frac{1}{2}\varphi \notin \Sigma$, and let $\mu \in \mathfrak{a}^*$. If a nonzero quotient of $X^{s_\varphi \mu}$ is isomorphic to a subquotient of X^μ, then $\mu(h_\varphi) \in \mathbf{Z}_+$ (the set of nonnegative integers).*

PROOF. Let x_0 be the canonical generator of X^μ and let $\mathfrak{N}^-$ be the universal enveloping algebra of $\coprod \mathfrak{g}^\psi$ ($\psi \in -\Sigma_+$). Since $X^\mu = \mathfrak{N}^- \cdot x_0$, X^μ is a direct sum of weight spaces for $\mathfrak{a}$ with weights of the form $\mu - \rho \sum n_\psi \psi$, where $n_\psi \in \mathbf{Z}_+$ and ψ ranges through the positive restricted roots. If a nonzero quotient Y of $X^{s_\varphi \mu}$ is isomorphic to a subquotient of X^μ, then any weight (for $\mathfrak{a}$) of Y must also be a weight of X^μ. Since $s_\varphi \mu - \rho = \mu - \rho - \mu(h_\varphi)\varphi$ is a weight of Y, it is also a weight of X^μ, and so $\mu(h_\varphi)\varphi$ must be a nonnegative integral linear combination of positive restricted roots. This implies that $\mu(h_\varphi) \in \mathbf{Z}_+$, by the last sentence of [1, 11.1.11]. Q.E.D.

If in the last proposition we assume that φ is simple, then we can say more (see Lemma 3.5). First we prove:

LEMMA 3.4. *Let $\varphi \in \Sigma_+$ such that $2\varphi \in \Sigma$, and let $\mu \in \mathfrak{a}^*$. If a nonzero quotient of $X^{s_\varphi \mu}$ is isomorphic to a subquotient of X^μ, then $\mu(h_\varphi) \in 2\mathbf{Z}_+$.*

PROOF. We may assume that k is algebraically closed. As in the proof of Proposition 3.3, we see that X^μ is a direct sum of weight spaces for $\mathfrak{h}$ with weights of the form $\mu - \rho - \sum n_\psi \psi$, where $n_\psi \in \mathbf{Z}_+$ and ψ ranges through R''_+. Also as in that proof, $s_\varphi \mu - \rho = \mu - \rho - \mu(h_\varphi)\varphi$, regarded as an element of $\mathfrak{h}^*$, is a weight of X^μ, and so $\mu(h_\varphi)\varphi$ is of the form $\sum n_\psi \psi$ ($n_\psi \in \mathbf{Z}_+, \psi \in R''_+$). But the fact that $2\varphi \in \Sigma_+$ implies that $2\varphi \in R''_+$; this follows from [5, p. 33, Lemmas 1 and 2], which are applicable by [2, Lemma 2.3] (cf. also [1, 1.14.14]). Thus $\mu(h_\varphi) \in 2\mathbf{Z}_+$ by the last sentence of [1, 11.1.11]. Q.E.D.

LEMMA 3.5. *Let $\alpha \in \Sigma_+$ be simple, and suppose* $\dim \mathfrak{g}^\alpha > 1$. *Also let $\mu \in \mathfrak{a}^*$. If a nonzero quotient of $X^{s_\alpha \mu}$ is isomorphic to a subquotient of X^μ, then $\mu(h_\alpha) \in 2\mathbf{Z}_+$.*

PROOF. Since the last lemma covers the case $2\alpha \in \Sigma$, we may assume that $2\alpha \notin \Sigma$. If a nonzero quotient of $X^{s_\alpha \mu}$ is isomorphic to a subquotient of X^μ, then X^μ must contain a (nonzero) $\mathfrak{m}$-invariant weight vector v for $\mathfrak{a}$ of weight $s_\alpha \mu - \rho = \mu - \rho - \mu(h_\alpha)\alpha$ (this follows easily from the fact that X^μ is a semisimple $\mathfrak{m} \oplus \mathfrak{a}$-module). By [3, Lemma 6.16], $v \in \mathfrak{N}^{\mathfrak{m}}_{-\alpha} \cdot x_0$, where x_0 is the

canonical generator of X^{μ}, $\mathfrak{U}_{-\alpha}$ is the universal enveloping algebra of $\mathfrak{g}^{-\alpha}$, and the superscript denotes $\mathfrak{m}$-invariants. But now by [3, Theorem 5.1, Case 2], v must have weight (for $\mathfrak{a}$) of the form $\mu - \rho - 2n\alpha$ ($n \in \mathbf{Z}_+$), and so $\mu(h_\alpha) \in 2\mathbf{Z}_+$. Q.E.D.

REMARK. Only the easy part of [3, Theorem 5.1], i.e., the part that does not require the Kostant-Mostow double transitivity theorem, was used here.

As in [3, §10], define $h'_\varphi \in \mathfrak{a}$ (where $\varphi \in \Sigma$) to be h_φ if $\dim \mathfrak{g}^\varphi > 1$ and $2h_\varphi$ if $\dim \mathfrak{g}^\varphi = 1$. Combining Theorems 2.1 and 3.2, Proposition 3.3 and Lemma 3.5, we have:

THEOREM 3.6. *Let $\alpha \in \Sigma_+$ be simple, and suppose $\lambda, \mu \in \mathfrak{a}^*$ are such that $\lambda - \mu$ is of the form $c\alpha$ for some $c \in k$. Then* $\dim \operatorname{Hom}_{\mathfrak{g}}(X^\lambda, X^\mu) \leqslant 1$, *and equality holds if and only if X^λ is isomorphic to a submodule of X^μ. If a nonzero quotient of X^λ is isomorphic to a subquotient of X^μ, then either $\lambda = \mu$, or else $\lambda = s_\alpha \mu$ and $\mu(h'_\alpha) \in 2\mathbf{Z}_+$.*

The following is now immediate:

COROLLARY 3.7. *Let α, λ and μ be as in the theorem. Let $S \subset X^\mu$ be the intersection of the conical space with the weight space for $\mathfrak{a}$ with weight $\lambda - \rho$. Then* $\dim S \leqslant 1$. *If* $\dim S = 1$, *then either $\lambda = \mu$, or else $\lambda = s_\alpha \mu$ and $\mu(h'_\alpha) \in 2\mathbf{Z}_+$.*

REMARK. This theorem and corollary include all the "uniqueness" assertions in Theorems 10.1 and 10.2 of [3].

Combining Theorem 3.6 with [3, Theorem 10.2], we get:

THEOREM 3.8. *Let α, λ and μ be as in Theorem* 3.6. *Then the following conditions are equivalent*:

(i) $\dim \operatorname{Hom}_{\mathfrak{g}}(X^\lambda, X^\mu) = 1$;

(ii) *X^λ is isomorphic to a submodule of X^μ*;

(iii) *a nonzero quotient of X^λ is isomorphic to a subquotient of X^μ*;

(iv) *either $\lambda = \mu$, or else $\lambda = s_\alpha \mu$ and $\mu(h'_\alpha) \in 2\mathbf{Z}_+$.*

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