PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 57, Number 2, June 1976

# A NOTE ON IDENTIFICATIONS OF METRIC SPACES

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ABSTRACT. A space $X$ is said to be $\sigma MK$ provided that $X$ has a countable closed cover $\mathcal{C}$ of metrizable subspaces such that if $K$ is a compact subset of $X$, there is a $C \in \mathcal{C}$ for which $K \subset C$. A Hausdorff space is $\sigma MK$ and Fréchet if and only if it is representable as a closed image of a metric space obtained by identifying a discrete collection of closed sets with hemicompact boundaries to points.

A familiar example of a nonmetrizable space is $R/N$, that is, the space obtained by identifying the set of natural numbers $N$, in the set of real numbers $R$, to a point and giving the resulting set the quotient topology. In [5], the concept of a $\sigma MK$ space proved useful in characterizing certain countably infinite spaces. This note relates identification spaces such as $R/N$ with the concept of a $\sigma MK$ space.

All spaces in this paper are understood to be Hausdorff topological spaces and all mappings are continuous onto functions. A space $X$ is $\sigma MK$ provided that $X$ has a countable closed cover $\mathcal{C}$ of metrizable subspaces such that if $K$ is a compact subset of $X$, there is a $C \in \mathcal{C}$ for which $K \subset C$. We may assume that $\mathcal{C}$ consists of sets $C_1 \subset C_2 \subset \cdots$, and we will henceforth do so. A space $X$ is *Fréchet* [2] provided that every accumulation point of a set $A$ in $X$ is the limit of some sequence in $A$. It is clear that $\sigma MK$ and Fréchet are each hereditary properties.

THEOREM 1. *If a space $X$ is $\sigma MK$ and Fréchet, then it is an image of a metric space $M$ under a closed mapping $f$, and there is a discrete collection $\mathfrak{F}$ of closed subsets of $M$, such that $f(F)$ is a point for each $F \in \mathfrak{F}$,* Bdy *$F$ is hemicompact for each $F \in \mathfrak{F}$, and $f$ is one-to-one upon restriction to $M - \bigcup \mathfrak{F}$.*

The proof follows from a number of propositions.

The concept of a $\sigma MK$ space arose in analogy to the concept of hemicompactness introduced by Arens [1]. A space $X$ is *hemicompact* provided that there is a countable cover $\mathcal{C}$ of compact subspaces such that if $K$ is a compact subset of $X$, there is a $C \in \mathcal{C}$ for which $K \subset C$. A metrizable space is hemicompact if and only if it is separable and locally compact.

PROPOSITION 2. (a) *If a space $X$ is $\sigma MK$ and Fréchet, then it is a closed image of a metric space having cardinality that of $X$.*

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Presented to the Society, November 7, 1975; received by the editors September 26, 1974 and, in revised form, May 18, 1975.

*AMS (MOS) subject classifications* (1970). Primary 54E20; Secondary 54C10.

*Key words and phrases.* $\sigma MK$ space, closed image of a metric space.

(b) *A space $X$ is hemicompact, Fréchet, and has every compact subspace metrizable if and only if it is a closed image of a locally compact separable metric space.* (*The metric space may be chosen to have cardinality that of $X$.*)

PROOF. We need only prove case (a), since case (b) is similar. (The "if" of case (b) is well known. In part, see [3].) Thus, assume that $X$ is Fréchet and $X = \bigcup\{C_n | n \in N\}$ as given in the definition of $\sigma MK$.

Let $C_0 = \varnothing$. Let $M_n = \mathrm{cl}\,(C_n - C_{n-1})$ for each $n$. Let $M$ be the discrete union of the $M_n$ and let $f$ be the natural mapping of $M$ onto $X$. Clearly $M$ is metrizable and $f$ is continuous. We need to show that $f$ is closed. Let $x_0$ be a point of $X$ for which there is a sequence $\{x_i\}$ of distinct points of $X - \{x_0\}$ converging to $x_0$. It suffices to show that if points $p_i \in f^{-1}(x_i)$ are chosen for each $i$, then the sequence $\{p_i\}$ has a convergent subsequence in the space $M$. So, let $p_i \in f^{-1}(x_i)$ for each $i$.

We will need the fact that there exists an integer $n_0$ for which $\{x_0, x_1, x_2, \ldots\}$ is contained in $\bigcup\{M_n | n = 1, 2, \ldots, n_0\}$, and $x_i \notin M_n$ for $i \in N$ and $n > n_0$. Suppose not. Then there exists a subsequence $\{x_{i_j}\}$ of $\{x_i\}$ and a subsequence $\{n_j\}$ of $N$ such that $x_{i_j} \in \mathrm{cl}\,(C_{n_j+1} - C_{n_j})$ for each $j$, with $n_j \neq n_k$ for distinct $j$, $k$. Then for each $j$, there is a sequence $\{x_k^j | k \in N\} \subset C_{n_j+1} - C_{n_j}$ such that $x_k^j \to x_{i_j}$. By the Fréchet assumption, since $x_0$ is an accumulation point of the set $\{x_k^l | k, l \in N\}$, there is a sequence $\{q_j\}$ contained in $\{x_k^l | k, l \in N\}$ such that $q_j \to x_0$ and $q_j \neq x_0$ for all $j$. Let $F$ be the set $\{q_j | j \in N\}$. Then for each $n$, $F \cap C_n$ is finite and so closed. Since $X$ is $\sigma MK$ and Fréchet, $F$ must be closed. This is impossible, so our supposition is false.

By the fact that we have just shown, there is a subsequence $\{x_{i_j}\}$ and there is an integer $n_1 \leqslant n_0$ for which $\{x_0, x_{i_1}, x_{i_2}, \ldots\}$ is contained in $M_{n_1}$, and for each $n > n_1$, $x_{i_j} \in M_n$ for at most finitely many $j \in N$. Thus there is an $n_2 \leqslant n_1$ for which a subsequence of $\{p_1, p_2, \ldots\}$ is contained in $M_{n_2}$. This subsequence of $\{p_1, p_2, \ldots\}$ converges in $M_{n_2}$, and so also in $M$.

LEMMA 3. *Suppose $S = \{0, i, (i,j,k) | i,j,k \in N\}$ has a topology with the following properties. Each point $(i,j,k)$ is itself an open set. Each set $S_i = \{i, (i,j,k) | j,k \in N\}$ is an open set and is homeomorphic to the "sequential fan" (that is, a set $G$ is a neighborhood of $i$ in $S_i$ if $i \in G$, and for each $j$, $(i,j,k) \in G$ for all but finitely many $k$). The sequence $\{i\}$ converges to* 0. *Then $S$ cannot be both $\sigma MK$ and Fréchet.*

PROOF. Suppose on the contrary that $S$ is Fréchet and $S = \bigcup\{C_i | i \in N\}$ as given by the definition of $\sigma MK$. We may assume (without loss of generality) that $\{0, 1, 2, \ldots\} \subset C_1$. Notice that for each $i$, $S_i$ is not contained in $C_i$, since $S_i$ is not metrizable. In fact, for each $i$, $S_i - C_i$ must contain a sequence $S'_i = \{(i,j,k_n) | n \in N\}$ for some $j \in N$ and some subsequence $\{k_n\}$ of $\{k | k \in N\}$. Let $S' = \bigcup\{S'_i | i \in N\}$. Then 0 is an accumulation point of $S'$. Since $S$ is assumed to be Fréchet, there is a sequence $T$ in $S'$ which converges to 0. But then, there is an integer $i_0$ for which $T \subset C_{i_0}$. Since $T \subset S'$ and $T$ converges to 0, there is a point $x_0$ common to $T$ and $\bigcup\{S'_i | i \geqslant i_0\}$. Let $i_1 \geqslant i_0$ be such that $x_0 \in T \cap S'_{i_1}$. Then $x_0 \in T \subset C_{i_0} \subset C_{i_1}$. Thus, $x_0 \in S'_{i_1} \cap C_{i_1}$. This is a contradiction.

PROPOSITION 4. *If a space $X$ is $\sigma MK$ and Fréchet, and $D$ is the set of those* *points of $X$ at which $X$ is not first-countable, then no point of $X$ is an accumulation point of $D$.*

PROOF. Otherwise, there exists a sequence $\{x_n\}$ of distinct points converging to a point $x_0 \in X$ such that each $x_n$(for $n = 1, 2, \ldots$) is a point of non-first-countability and $x_n \neq x_0$. There exists a sequence of disjoint open sets $G_n$ such that $x_n \in G_n$ for $n = 1, 2, \ldots$. Since each $G_n$ is Fréchet, but not countably bisequential at $x_n$ (since a closed image of a metric space which is countably bisequential is metrizable), by [6], there exists a copy of the sequential fan in $G_n$ "at $x_n$". Let $S_n$ denote this copy. Thus, for each $n = 1, 2, \ldots$, there is an $S_n$ "at $x_n$" and these $S_n$ are disjoint. Let $S = \bigcup\{S_n | n = 1, 2, \ldots\} \cup \{x_0\}$. By Lemma 3, $S$ is either not $\sigma MK$ or not Fréchet. We have a contradiction.

PROPOSITION 5. *If a space $X = M/F$, where $X$ is $\sigma MK$, $M$ is metrizable, and $F$ is a closed subset of $M$, then* Bdy *$F$ is hemicompact.*

PROOF. If Bdy $F$ is empty, it is trivially hemicompact. If Bdy $F$ is nonempty, $M/F = (M - \text{Int } F)/\text{Bdy } F$, so we may assume (without loss of generality) that the interior of $F$ is empty. Let $X = \bigcup\{C_n | n \in N\}$ as given by the definition of $\sigma MK$. Let $f$ be the natural mapping of $M$ into $X$. Since there exists an $n \in N$ for which the point $f(F) \in C_n$, we may assume that in fact, $f(F) \in C_1$. Since $C_n$ is metrizable, $f_n = f|f^{-1}(C_n)$ is a closed mapping of $f^{-1}(C_n)$ onto $C_n$ with $\text{Bdy} f_n^{-1}(x)$ being compact for each point $x$ of $C_n$ ([4] or [7]). Let $K_n = \text{Bdy } f_n^{-1}(f(F))$ for each $n$, where the boundary is taken relative to $f^{-1}(C_n)$. Then each $K_n$ is compact. Also, $F = \bigcup\{K_n | n \in N\}$. Because, if $p \in F$, there is a sequence $\{p_i\}$ in $M - F$ which converges to $p$. But there is an integer $n_0$ for which $\{f(p), f(p_1), f(p_2), \ldots\} \subset C_{n_0}$. Thus, $p \in \text{cl } (f^{-1}(C_{n_0}) - F) \cap F = K_{n_0}$.

In order to show that $\{K_n | n \in N\}$ is a sequence as in the definition of hemicompact, let $K$ be a compact subset of $F$. Suppose that for all $n$, there is a point $x_n \in K - K_n \subset K - \text{cl } (f^{-1}(C_n) - F)$. Since $K$ is sequentially compact, there is a point $x$ in $K$ which is the limit of some subsequence of $\{x_n\}$. For simplicity of notation, assume that the subsequence is $\{x_n\}$ itself. Let $\{G_n | n \in N\}$ be an open base at $x$ in $M$ such that $G_n \supset G_{n+1}$ for all $n$. We may also assume that $x_n \in G_n$ for all $n$. Since $x_n$ is in (the boundary of) $F$, there exists a sequence $\{x_m^n | m \in N\}$ in $G_n - F$ which converges to $x_n$. Let $L = \{x, x_n, x_m^n | m, n \in N\}$. Then $L$ is a compact subset of $M$. Since $f(L)$ is compact in $X$, there is an integer $n_0$ for which $f(L) \subset C_{n_0}$. Then the sequence $\{x_m^{n_0} | m \in N\}$ is contained in $f^{-1}(C_{n_0}) - F$. This means that $x_{n_0} \in \text{cl } (f^{-1}(C_{n_0}) - F)$. We have a contradiction, and thereby we have shown that (the boundary of) $F$ is hemicompact.

PROOF OF THEOREM 1. Let $X$ be $\sigma MK$ and Fréchet. By Proposition 2, $X$ is a closed image of a metric space $M'$ under a mapping $h$. Let $D$ be the set of those points of $X$ at which $X$ is not first-countable. By Proposition 4, $D$ is a discrete closed subspace of $X$, and ([4] or [7]) $X - D$ is metrizable. Let $M = h^{-1}(D) \cup (X - D)$, define $g\colon M' \to M$ by $g|h^{-1}(D) =$ identity, $g|h^{-1}(X - D) = h|h^{-1}(X - D)$, and $f\colon M \to X$ by $f|h^{-1}(D) = h|h^{-1}(D)$, $f|(X - D) =$ identity. Giving $M$ the quotient topology as an image of $M'$, $g$

and $f$ are closed continuous mappings and $M$ is metrizable. Let $\mathfrak{F} = \{f^{-1}(x) | x \in D\}$. Then $\mathfrak{F}$ is a discrete collection in $M$ and by Proposition 5, each $F \in \mathfrak{F}$ has a hemicompact boundary. The proof is complete.

We now prove the converse of Theorem 1.

THEOREM 6. *If a space $X$ is an image of a metric space $M$ under a closed mapping $f$, and there is a discrete collection $\mathfrak{F}$ of closed subsets of $M$, such that $f(F)$ is a point for each $F \in \mathfrak{F}$,* Bdy *$F$ is hemicompact for each $F \in \mathfrak{F}$, and $f$ is one-to-one upon restriction to $M - \bigcup \mathfrak{F}$, then $X$ is $\sigma MK$ and Fréchet.*

PROOF. It is well known that a closed image of a metric space is Fréchet. We prove that $X$ is $\sigma MK$ under the stated hypotheses. We may assume (without loss of generality) that Int $F = \varnothing$ for each $F \in \mathfrak{F}$. Let each $F = \bigcup \{C_n^F | n \in N\}$ as given by the definition of hemicompact, and we may assume that each $C_n^F \subset C_{n+1}^F$. Since $\mathfrak{F}$ is a discrete collection in $M$, there exists a discrete collection $\{G^F | F \in \mathfrak{F}\}$ of open sets such that $F \subset G^F$ for every $F \in \mathfrak{F}$. Also, let $G_n^F = G^F \cap S_{1/n}(F - C_n^F)$, where $S_{1/n}(A)$ denotes the $1/n$ open sphere around set $A$. Let $D_n^F = C_n^F - G_n^F$. Finally, let $M'_n = M - \bigcup \{G_n^F | F \in \mathfrak{F}\}, f_n = f | M'_n$, and $M_n = f_n(M'_n)$ for all $n$. Since $f$ is a closed mapping and $M'_n$ is a closed subset of $M$, each $f_n$ is a closed mapping. Also, $f_n^{-1}(x)$ is compact for each point $x$ of $X$, since each $D_n^F$ is compact. By [4] or [7], each $M_n$ is then metrizable. It is clear that $X = \bigcup \{M_n | n \in N\}$.

Now let $K$ be a compact subset of $X$. And suppose that $K \not\subset M_n$ for all $n$. Then for each $n$, there is a point $x_n \in K - M_n$. But $K$ is sequentially compact (being compact and Fréchet), and so there is a subsequence $\{x_{n_i}\}$ of distinct points, converging to a point $x$ of $K$, with $x_{n_i} \neq x$ for all $i$. Fix $m \in N$. Since the sequence $\{x_{n_i}\}$ meets $M_m$ in at most finitely many points, there exists an $i_m \in N$ such that $x_{n_i} \in f(\bigcup \{G_m^F | F \in \mathfrak{F}\})$ for all $i > i_m$. For each $i > i_m$, let $y_{n_i} \in f^{-1}(x_{n_i}) \cap \bigcup \{G_m^F | F \in \mathfrak{F}\}$. Since the set $\{y_{n_i} | i > i_m\}$ is not closed in $M$, let $y$ be an accumulation point of this set. So there exists a subsequence of $\{y_{n_i}\}$ which converges to $y$. This means that $f(y) = x$. Also, $y \in \text{cl} \bigcup \{G_m^F | F \in \mathfrak{F}\} \subset \bigcup \{G_{m-1}^F | F \in \mathfrak{F}\}$ for all $m$. We then have that $y \in F_0$ for some $F_0 \in \mathfrak{F}$. Since $y \in G_m^{F_0} \subset S_{1/m}(F_0 - C_m^{F_0})$ for every $m$, let $p_m$ be a point of $F_0 - C_m^{F_0}$ for which $d(y, p_m) < 1/m$. Then $p_m \to y$ in $F_0$. So there is an integer $n_0$ for which $\{y, p_1, p_2, \ldots\} \subset C_{n_0}^{F_0}$. Thus $p_{n_0} \in C_{n_0}^{F_0}$. This is a contradiction.

EXAMPLES 7. To illustrate the results of this paper we consider two examples of countable regular Fréchet spaces. Let $Q$ denote the usual space of rational numbers. Let $Q'$ be $Q^2$ in the plane together with the entire $x$-axis of real numbers, with the usual topology from the plane. Let $X_1$ be the quotient space obtained by considering $Q'$ and identifying the $x$-axis to a point. Let $X_2$ be the quotient space obtained by considering $Q^2$ and identifying the set $Q$ in the $x$-axis to a point. Then $X_1$ and $X_2$ are the desired spaces. By Theorem 6, $X_1$ is $\sigma MK$. On the other hand, $X_2$ is not $\sigma MK$. To see this, suppose $X_2 = \bigcup \{C_n | n \in N\}$ as given by the definition of $\sigma MK$ and suppose that $C_n \subset C_{n+1}$ for all $n$. Since $X_2$ is not metrizable, for each $n$ there exists a point $x_n$ in $[([0, 1/n]^2 \cap Q^2)/Q] - C_n$. Then $x_n \to (0, 0)$. So there exists an integer $n_0$ for which $\{x_n | n \in N\} \subset C_{n_0}$, contradiction.

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