A NOTE ON IDENTIFICATIONS OF METRIC SPACES

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ABSTRACT. A space \( X \) is said to be \( \sigma MK \) provided that \( X \) has a countable closed cover \( \mathcal{C} \) of metrizable subspaces such that if \( K \) is a compact subset of \( X \), there is a \( C \in \mathcal{C} \) for which \( K \subseteq C \). A Hausdorff space is \( \sigma MK \) and Fréchet if and only if it is representable as a closed image of a metric space obtained by identifying a discrete collection of closed sets with hemicompact boundaries to points.

A familiar example of a nonmetrizable space is \( R/N \), that is, the space obtained by identifying the set of natural numbers \( N \), in the set of real numbers \( R \), to a point and giving the resulting set the quotient topology. In [5], the concept of a \( \sigma MK \) space proved useful in characterizing certain countably infinite spaces. This note relates identification spaces such as \( R/N \) with the concept of a \( \sigma MK \) space.

All spaces in this paper are understood to be Hausdorff topological spaces and all mappings are continuous onto functions. A space \( X \) is \( \sigma MK \) provided that \( X \) has a countable closed cover \( \mathcal{C} \) of metrizable subspaces such that if \( K \) is a compact subset of \( X \), there is a \( C \in \mathcal{C} \) for which \( K \subseteq C \). We may assume that \( \mathcal{C} \) consists of sets \( C_1 \subseteq C_2 \subseteq \cdots \), and we will henceforth do so. A space \( X \) is Fréchet [2] provided that every accumulation point of a set \( A \) in \( X \) is the limit of some sequence in \( A \). It is clear that \( \sigma MK \) and Fréchet are each hereditary properties.

THEOREM 1. If a space \( X \) is \( \sigma MK \) and Fréchet, then it is an image of a metric space \( M \) under a closed mapping \( f \), and there is a discrete collection \( \mathcal{F} \) of closed subsets of \( M \), such that \( f(F) \) is a point for each \( F \in \mathcal{F} \), \( \text{Bdy} F \) is hemicompact for each \( F \in \mathcal{F} \), and \( f \) is one-to-one upon restriction to \( M - \bigcup \mathcal{F} \).

The proof follows from a number of propositions.

The concept of a \( \sigma MK \) space arose in analogy to the concept of hemicompactness introduced by Arens [1]. A space \( X \) is hemicompact provided that there is a countable cover \( \mathcal{C} \) of compact subspaces such that if \( K \) is a compact subset of \( X \), there is a \( C \in \mathcal{C} \) for which \( K \subseteq C \). A metrizable space is hemicompact if and only if it is separable and locally compact.

PROPOSITION 2. (a) If a space \( X \) is \( \sigma MK \) and Fréchet, then it is a closed image of a metric space having cardinality that of \( X \).
(b) A space $X$ is hemicompact, Fréchet, and has every compact subspace metrizable if and only if it is a closed image of a locally compact separable metric space. (The metric space may be chosen to have cardinality that of $X$.)

Proof. We need only prove case (a), since case (b) is similar. (The “if” of case (b) is well known. In part, see [3].) Thus, assume that $X$ is Fréchet and $X = \bigcup \{C_n \mid n \in \mathbb{N}\}$ as given in the definition of $\sigma MK$. Let $C_0 = \emptyset$. Let $M_n = \text{cl} (C_n - C_{n-1})$ for each $n$. Let $M$ be the discrete union of the $M_n$ and let $f$ be the natural mapping of $M$ onto $X$. Clearly $M$ is metrizable and $f$ is continuous. We need to show that $f$ is closed. Let $x_0$ be a point of $X$ for which there is a sequence $\{x_i\}$ of distinct points of $X - \{x_0\}$ converging to $x_0$. It suffices to show that if points $p_i \in f^{-1}(x_i)$ are chosen for each $i$, then the sequence $\{p_i\}$ has a convergent subsequence in the space $M$. So, let $p_i \in f^{-1}(x_i)$ for each $i$.

We will need the fact that there exists an integer $n_0$ for which $\{x_0, x_1, x_2, \ldots\}$ is contained in $\bigcup \{M_n \mid n = 1, 2, \ldots, n_0\}$, and $x_i \notin M_n$ for $i \in \mathbb{N}$ and $n > n_0$. Suppose not. Then there exists a subsequence $\{x_{i_n}\}$ of $\{x_i\}$ and a subsequence $\{n_j\}$ of $\mathbb{N}$ such that $x_{i_j} \notin \text{cl} (C_{n_j+1} - C_{n_j})$ for each $j$, with $n_j \neq n_k$ for distinct $j, k$. Then for each $i$, there is a sequence $\{x_{k_i} \mid k \in \mathbb{N}\} \subset C_{n_i+1} - C_{n_i}$ such that $x_{k_i} \to x_i$. By the Fréchet assumption, since $x_0$ is an accumulation point of the set $\{x_{k_i} \mid k \in \mathbb{N}\}$, there is a sequence $\{q_j\}$ contained in $\{x_{k_i} \mid k \in \mathbb{N}\}$ such that $q_j \to x_0$ and $q_j \neq x_0$ for all $j$. Let $F$ be the set $\{q_j \mid j \in \mathbb{N}\}$. Then for each $n, F \cap C_n$ is finite and so closed. Since $X$ is $\sigma MK$ and Fréchet, $F$ must be closed. This is impossible, so our supposition is false.

By the fact that we have just shown, there is a subsequence $\{x_{i_j}\}$ and there is an integer $n_1 \leq n_0$ for which $\{x_0, x_{i_1}, x_{i_2}, \ldots\}$ is contained in $M_{n_1}$, and for each $n > n_1$, $x_i \in M_n$ for at most finitely many $j \in \mathbb{N}$. Thus there is an $n_2 \leq n_1$ for which a subsequence of $\{p_1, p_2, \ldots\}$ is contained in $M_{n_2}$. This subsequence of $\{p_1, p_2, \ldots\}$ converges in $M_{n_2}$, and so also in $M$.

Lemma 3. Suppose $S = \{0, i, (i,j,k) \mid i, j, k \in \mathbb{N}\}$ has a topology with the following properties. Each point $(i,j,k)$ is itself an open set. Each set $S_i = \{i, (i,j,k) \mid j, k \in \mathbb{N}\}$ is an open set and is homeomorphic to the “sequential fan” (that is, a set $G$ is a neighborhood of $i$ in $S_i$ if $i \in G$, and for each $j$, $(i,j,k) \in G$ for all but finitely many $k$). The sequence $\{i\}$ converges to 0. Then $S$ cannot be both $\sigma MK$ and Fréchet.

Proof. Suppose on the contrary that $S$ is Fréchet and $S = \bigcup \{C_i \mid i \in \mathbb{N}\}$ as given by the definition of $\sigma MK$. We may assume (without loss of generality) that $\{0,1,2, \ldots\} \subset C_1$. Notice that for each $i$, $S_i$ is not contained in $C_i$, since $S_i$ is not metrizable. In fact, for each $i$, $S_i - C_i$ must contain a sequence $S_i' = \{(i,j,k) \mid n \in \mathbb{N}\}$ for some $j \in \mathbb{N}$ and some subsequence $\{k_n\}$ of $\{k \mid k \in \mathbb{N}\}$. Let $S' = \bigcup \{S_i' \mid i \in \mathbb{N}\}$. Then 0 is an accumulation point of $S'$. Since $S$ is assumed to be Fréchet, there is a sequence $T$ in $S'$ which converges to 0. But then, there is an integer $i_0$ for which $T \subset C_{i_0}$. Since $T \subset S'$ and $T$ converges to 0, there is a point $x_0$ common to $T$ and $\bigcup \{S_i' \mid i \geq i_0\}$. Let $i_1 \geq i_0$ be such that $x_0 \in T \cap S_{i_1}'$. Then $x_0 \in T \subset C_{i_0} \subset C_{i_1}$. Thus, $x_0 \in S_{i_1}' \subset C_{i_1}$. This is a contradiction.

Proposition 4. If a space $X$ is $\sigma MK$ and Fréchet, and $D$ is the set of those
points of \( X \) at which \( X \) is not first-countable, then no point of \( X \) is an accumulation point of \( D \).

**Proof.** Otherwise, there exists a sequence \( \{x_n\} \) of distinct points converging to a point \( x_0 \in X \) such that each \( x_n \) (for \( n = 1, 2, \ldots \)) is a point of non-first-countability and \( x_n \neq x_0 \). There exists a sequence of disjoint open sets \( G_n \) such that \( x_n \in G_n \) for \( n = 1, 2, \ldots \). Since each \( G_n \) is Fréchet, but not countably bisequential at \( x_n \) (since a closed image of a metric space which is countably bisequential is metrizable), by [6], there exists a copy of the sequential fan in \( G_n \) "at \( x_n \). Let \( S_n \) denote this copy. Thus, for each \( n = 1, 2, \ldots \), there is an \( S_n \) "at \( x_n \)" and these \( S_n \) are disjoint. Let \( S = \cup \{ S_n | n = 1, 2, \ldots \} \cup \{ x_0 \} \). By Lemma 3, \( S \) is either not \( \sigma MK \) or not Fréchet. We have a contradiction.

**Proposition 5.** If a space \( X = M/F \), where \( X \) is \( \sigma MK \), \( M \) is metrizable, and \( F \) is a closed subset of \( M \), then \( Bdy \ F \) is hemicompact.

**Proof.** If \( Bdy \ F \) is empty, it is trivially hemicompact. If \( Bdy \ F \) is nonempty, \( M/F = (M - Int F)/Bdy F \), so we may assume (without loss of generality) that the interior of \( F \) is empty. Let \( X = \cup \{ C_n | n \in N \} \) as given by the definition of \( \sigma MK \). Let \( f \) be the natural mapping of \( M \) into \( X \). Since there exists an \( n \in N \) for which the point \( f(F) \in C_n \), we may assume that in fact, \( f(F) \in C_1 \). Since \( C_n \) is metrizable, \( f_n = f|f^{-1}(C_n) \) is a closed mapping of \( f^{-1}(C_n) \) onto \( C_n \) with \( Bdy f_n^{-1}(x) \) being compact for each point \( x \) of \( C_n \) ([4] or [7]). Let \( K_n = Bdy f_n^{-1}(f(F)) \) for each \( n \), where the boundary is taken relative to \( f^{-1}(C_n) \). Then each \( K_n \) is compact. Also, \( F = \cup \{ K_n | n \in N \} \). Because, if \( p \in F \), there is a sequence \( \{ p_n \} \) in \( M - F \) which converges to \( p \). But there is an integer \( n_0 \) for which \( \{ f(p), f(p_1), f(p_2), \ldots \} \subset C_{n_0} \). Thus, \( p \in cl(f^{-1}(C_{n_0}) - F) \). We have a contradiction, and thereby we have shown that \( (\text{the boundary of}) \ F \) is hemicompact.

**Proof of Theorem 1.** Let \( X \) be \( \sigma MK \) and Fréchet. By Proposition 2, \( X \) is a closed image of a metric space \( M' \) under a mapping \( h \). Let \( D \) be the set of those points of \( X \) at which \( X \) is not first-countable. By Proposition 4, \( D \) is a discrete closed subspace of \( X \), and (for \([4]\) or [7]) \( X - D \) is metrizable. Let \( M = h^{-1}(D) \cup (X - D) \), define \( g: M' \to M \) by \( g|h^{-1}(D) = \text{identity} \), \( g|X - D = h|X - D \), and \( f: M \to X \) by \( f|h^{-1}(D) = h|h^{-1}(D) \), \( f|(X - D) = \text{identity} \). Giving \( M \) the quotient topology as an image of \( M' \), \( g \)
and $f$ are closed continuous mappings and $M$ is metrizable. Let $\mathcal{T} = \{ f^{-1}(x) \mid x \in D \}$. Then $\mathcal{T}$ is a discrete collection in $M$ and by Proposition 5, each $F \in \mathcal{T}$ has a hemicompact boundary. The proof is complete.

We now prove the converse of Theorem 1.

**Theorem 6.** If a space $X$ is an image of a metric space $M$ under a closed mapping $f$, and there is a discrete collection $\mathcal{T}$ of closed subsets of $M$, such that $f(F)$ is a point for each $F \in \mathcal{T}$, $\text{Bdy} F$ is hemicompact for each $F \in \mathcal{T}$, and $f$ is one-to-one upon restriction to $M - \bigcup \mathcal{T}$, then $X$ is $\sigma MK$ and Fréchet.

**Proof.** It is well known that a closed image of a metric space is Fréchet. We prove that $X$ is $\sigma MK$ under the stated hypotheses. We may assume (without loss of generality) that $\text{Int} F = \emptyset$ for each $F \in \mathcal{T}$. Let each $F = \bigcup \{ C_n^F \mid n \in N \}$ as given by the definition of hemicompact, and we may assume that each $C_n^F \subset C_{n+1}^F$. Since $\mathcal{T}$ is a discrete collection in $M$, there exists a discrete collection $\{ G_n^F \mid F \in \mathcal{T} \}$ of open sets such that $F \subset G_n^F$ for every $F \in \mathcal{T}$. Also, let $G_n^F = G_n^F \cap S_{1/n}(F - C_n^F)$, where $S_{1/n}(A)$ denotes the $1/n$ open sphere around set $A$. Let $D_n^F = C_n^F - G_n^F$. Finally, let $M'_n = M - \bigcup \{ G_n^F \mid F \in \mathcal{T} \}$. Then $f_n = f | M'_n$, and $M_n = f_n(M'_n)$ for all $n$. Since $f$ is a closed mapping and $M_n$ is a closed subset of $M$, each $f_n$ is a closed mapping. Also, $f_n^{-1}(x)$ is compact for each point $x$ of $X$, since each $D_n^F$ is compact. By [4] or [7], each $M_n$ is then metrizable. It is clear that $X = \bigcup \{ M_n \mid n \in N \}$.

Now let $K$ be a compact subset of $X$. And suppose that $K \subset M_n$ for all $n$. Then for each $n$, there is a point $x_n \in K - M_n$. But $K$ is sequentially compact (being compact and Fréchet), and so there is a subsequence $(x_{n_i})$ of distinct points, converging to a point $x$ of $K$, with $x_{n_i} \neq x$ for all $i$. Fix $m \in N$. Since the sequence $(x_{n_i})$ meets $M_m$ in at most finitely many points, there exists an $i_m \in N$ such that $x_{n_i} \in f(\{ G_n^F \mid F \in \mathcal{T} \})$ for all $i > i_m$. For each $i > i_m$, let $y_{n_i} \in f^{-1}(x_{n_i}) \cap \{ G_n^F \mid F \in \mathcal{T} \}$. Since the set $(y_{n_i} \mid i > i_m)$ is not closed in $M$, let $y$ be an accumulation point of this set. So there exists a subsequence of $(y_{n_i})$ which converges to $y$. This means that $f(y) = x$. Also, $y \in l(\bigcup \{ G_n^F \mid F \in \mathcal{T} \} \subset \{ G_{m-1}^F \mid F \in \mathcal{T} \}$ for all $m$. We then have that $y \in F_0$ for some $F_0 \in \mathcal{T}$. Since $y \in G_{m_0}^F \subset S_{1/m}(F_0 - C_{m_0}^F)$ for each $m$, let $p_m$ be a point of $F_0 - C_{m_0}^F$ for which $d(y, p_m) < 1/m$. Then $p_m \to y$ in $F_0$. So there is an integer $n_0$ for which $(y, p_1, p_2, \ldots) \subset C_{n_0}^F$. Thus $p_{n_0} \in C_{n_0}^F$. This is a contradiction.

**Examples 7.** To illustrate the results of this paper we consider two examples of countable regular Fréchet spaces. Let $Q$ denote the usual space of rational numbers, and $Q^2$ be $Q^2$ in the plane together with the entire x-axis of real numbers, with the usual topology from the plane. Let $X_1$ be the quotient space obtained by considering $Q^2$ and identifying the x-axis to a point. Let $X_2$ be the quotient space obtained by considering $Q^2$ and identifying the set $Q$ in the x-axis to a point. Then $X_1$ and $X_2$ are the desired spaces. By Theorem 6, $X_1$ is $\sigma MK$. On the other hand, $X_2$ is not $\sigma MK$. To see this, suppose $X_2 = \bigcup \{ C_n \mid n \in N \}$ as given by the definition of $\sigma MK$ and suppose that $C_n \subset C_{n+1}$ for all $n$. Since $X_2$ is not metrizable, for each $n$ there exists a point $x_n$ in $[(0, 0), (1, 0)] \cap Q^2$. Then $x_n \to (0, 0)$. So there exists an integer $n_0$ for which $(x_n \mid n \in N) \subset C_{n_0}$, contradiction.
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