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INEQUALITIES FOR POLYNOMIALS SATISFYING $p(z) \equiv z^n \bar{p}(1/z)$

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ABSTRACT. If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n, then it is known that $\text{Max}_{|z|=1}|p'(z)| \leqslant n\ \text{Max}_{|z|=1}|p(z)|$. In this paper we obtain the analogous inequality for a subclass of polynomials satisfying $p(z) \equiv z^n \bar{p}(1/z)$. Some other inequalities have also been obtained.

1. **Introduction and statement of results.** Let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n and $p'(z)$ its derivative. Concerning the estimate of $|p'(z)|$ on the unit disc $|z| \leqslant 1$, we have the following theorem of Bernstein [2].

THEOREM A. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n such that $|p(z)| \leqslant 1$ for $|z| \leqslant 1$ then*

$$|p'(z)| \leqslant n \quad \text{for } |z| = 1. \tag{1.1}$$

This result is best possible and equality holds for $p(z) = \alpha z^n$, where $|\alpha| = 1$.

On the other hand, concerning the estimate of $|p(z)|$ on the disc $|z| = R > 1$, we have the following theorem, which is a simple consequence of the maximum modulus principle.

THEOREM B. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n, then*

$$|p(Re^{i\theta})| \leqslant R^n \max_{0 \leqslant \theta < 2\pi} |p(e^{i\theta})|, \qquad R \geqslant 1, \theta \text{ real}. \tag{1.2}$$

This result is best possible and equality holds for $p(z) = \alpha z^n$.

For polynomials having no zero in $|z| < 1$, an inequality analogous to (1.1) was obtained by Lax [5] and analogous to (1.2) by Ankeny and Rivlin [1]. Malik [6] (see also Govil and Rahman [4]) considered the class of polynomials having no zero in $|z| < k$, $k \geqslant 1$, and obtained an inequality analogous to (1.1). The class of polynomials having a prescribed zero on $|z| = 1$ has been considered by Giroux and Rahman [3]. It was proposed by Professor Q. I. Rahman to study the class of polynomials satisfying $p(z) \equiv z^n \bar{p}(1/z)$ and obtain inequalities analogous to (1.1) and (1.2).

While trying to solve the problem proposed by Professor Rahman, we have been able to obtain inequalities analogous to (1.1) and (1.2) for the class of polynomials satisfying $p(z) \equiv z^n \bar{p}(1/z)$ and having all the zeros either in the left half-plane or in the right half-plane. Throughout this paper, we shall denote by Π_n, the class of polynomials $p(z)$ of degree n satisfying $p(z)$

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$\equiv z^n p(1/z)$ and having all their zeros either in the left half-plane or in the right half-plane. We prove

THEOREM 1. *If $p(z)$ is a polynomial belonging to the class Π_n, then*

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{2^{1/2}} \max_{|z|=1} |p(z)| \tag{1.3}$$

and

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.4}$$

Inequality (1.4) *is best possible and equality holds for the polynomial $p(z) = (1+z)^n$ when the zeros lie in the left half-plane and for polynomial $p(z) = (1-z)^n$ when zeros lie in the right half-plane.*

We do not know if inequality (1.3) is best possible. However, for the class of polynomials satisfying only $p(z) \equiv z^n p(1/z)$, the bound in inequality (1.3) cannot be smaller than $n/2^{1/2}$, as is evident from the example $p(z) = z^n + 2iz^{n/2} + 1$, n being even.

If we use inequality (1.3), the argument of Ankeny and Rivlin [1] yields the following

THEOREM 2. *If $p(z)$ is a polynomial belonging to class Π_n, then*

$$\max_{|z|=R>1} |p(z)| \leqslant \frac{R^n + (2^{1/2}-1)}{2^{1/2}} \max_{|z|=1} |p(z)|. \tag{1.5}$$

2. Lemmas.

LEMMA 1. *For any polynomial $p(z) = (z-a)(z-1/a)$, $a \neq 0$ and $|\arg a| \geqslant \pi/2$ we have*

$$\max_{|z|=1} |p(z)| = |p(1)|. \tag{2.1}$$

If we replace the hypothesis $|\arg a| \geqslant \pi/2$ by $|\arg a| \leqslant \pi/2$, then

$$\max_{|z|=1} |p(z)| = |p(-1)|. \tag{2.2}$$

PROOF OF LEMMA 1. We can assume without loss of generality, $0 < |a| \leqslant 1$. For if $|a| > 1$, apply the result with $1/a$ instead of a. Let $a = re^{i\varphi}$, $0 < r \leqslant 1$, $|\varphi| \geqslant \pi/2$. Then for $|z| = 1$, we have

$$\begin{aligned} |p(z)| &= |p(e^{i\theta})| = |(e^{i\theta} - re^{i\varphi})(e^{i\theta} - r^{-1}e^{-i\varphi})| \\ &= |e^{2i\theta} + 1 - e^{i\theta}(re^{i\varphi} + r^{-1}e^{-i\varphi})| \\ &= |e^{i\theta} + e^{-i\theta} - (re^{i\varphi} + r^{-1}e^{-i\varphi})| = |2\cos\theta - 2\omega(r,\varphi)| \end{aligned} \tag{2.3}$$

where

$$2\omega(r,\varphi) = re^{i\varphi} + r^{-1}e^{-i\varphi}. \tag{2.4}$$

Now since $|\varphi| \geqslant \pi/2$, the real part of $\omega(r,\varphi)$ is nonpositive and so the right-hand side of (2.3) will be maximum when $\cos\theta$ is maximum, i.e. when $\theta = 0$. Therefore we have

$$\max_{|z|=1} |p(z)| = |2 - 2\omega(r,\varphi)| = |p(1)|.$$

The proof of (2.2) follows on the same lines as of (2.1), so we omit it.

LEMMA 2. *For any complex number z_0 lying in the closed left half-plane* $(1 + |z_0|)/|1 - z_0| \leq 2^{1/2}$.

PROOF OF LEMMA 2. For any complex number z_0 we have

$$\frac{1+|z_0|}{|1-z_0|} = \frac{1+|z_0|}{\left\{1+|z_0|^2 - 2\text{Re } z_0\right\}^{1/2}} \leq \frac{1+|z_0|}{\left\{1+|z_0|^2\right\}^{1/2}} \tag{2.5}$$

because z_0 lies in the closed left half-plane. So from (2.5) we have

$$(1+|z_0|)/|1-z_0| \leq \left\{1 + 2|z_0|/\left(1+|z_0|^2\right)\right\}^{1/2} \leq 2^{1/2}.$$

3. **Proofs of the theorems.**

PROOF OF THEOREM 1. Let the polynomial $p(z)$ belonging to the class Π_n be $\sum_{\nu=0}^{n} a_\nu z^\nu$. We shall prove the result when $p(z)$ has all its zeros in the closed left half-plane. In case $p(z)$ has all its zeros in the closed right half-plane, the result follows on the same lines and by using (2.2) instead of (2.1). Firstly, note that $p(z)$ cannot have a zero at the origin; for if $p(z)$ has a zero at the origin, then $a_0 = 0$ and since $p(z)$ satisfies $p(z) \equiv z^n p(1/z)$, we have $a_n = 0$,which implies that $p(z)$ is of degree less than n. Further, again on account of hypothesis $p(z) \equiv z^n p(1/z)$, if $re^{i\alpha}$ is a zero of $p(z)$ then $r^{-1}e^{-i\alpha}$ is also a zero of $p(z)$ and hence the zeros of $p(z)$ occur in pairs at $z = re^{i\alpha}$ and $z = r^{-1}e^{-i\alpha}$. If $r = 1$ and $\alpha = \pi$, then $re^{i\alpha} = r^{-1}e^{-i\alpha}$ and hence the zeros of $p(z)$ occur either in pairs at $z = re^{i\alpha}$ and $z = r^{-1}e^{-i\alpha}$ or at $z = -1$. Thus the polynomial $p(z)$ is of the form

$$p(z) = (z+1)^m \left[\prod_{k=1}^{l} \left\{ \left(z - r_k e^{i\alpha_k}\right)\left(z - \frac{1}{r_k} e^{-i\alpha_k}\right) \right\} \right] \tag{3.1}$$

where $m \geq 0$; $m + 2l = n$; $r_k > 0$ and $|\alpha_k| \geq \pi/2$ for $k = 1, 2, \ldots, l$. Then

$$\underset{|z|=1}{\text{Max}} |p(z)| = \underset{|z|=1}{\text{Max}} \left[|z+1|^m \prod_{k=1}^{l} \left| \left\{ \left(z - r_k e^{i\alpha_k}\right)\left(z - \frac{1}{r_k} e^{-i\alpha_k}\right) \right\} \right| \right]$$

$$= |1+1|^m \prod_{k=1}^{l} \left| \left\{ \left(1 - r_k e^{i\alpha_k}\right)\left(1 - \frac{1}{r_k} e^{-i\alpha_k}\right) \right\} \right| \quad \text{(by (2.1))} \tag{3.2}$$

$$= |p(1)| \tag{3.3}$$

$$= 2^m \prod_{k=1}^{l} |2 - 2\omega(r_k, \alpha_k)| \quad \text{(by (3.2) and (2.4))}$$

$$= 2^{m+1} \prod_{k=1}^{l} |1 - \omega(r_k, \alpha_k)|. \tag{3.4}$$

Also

$$p(z) = (z+1)^m \left\{ \prod_{k=1}^{l} \left[z^2 - 2\omega(r_k, \alpha_k) z + 1 \right] \right\}$$

and

$$p'(z) = m(z+1)^{m-1} \prod_{k=1}^{l} \left[z^2 - 2\omega(r_k, \alpha_k)z + 1\right] \tag{3.5}$$
$$+ (z+1)^m \sum_{k=1}^{l} \left\{ [2z - 2\omega(r_k, \alpha_k)] \prod_{j=1; j\neq k}^{l} \left[z^2 - 2\omega(r_j, \alpha_j)z + 1\right] \right\}.$$

Therefore for $|z| = 1$,

$$|p'(z)| \leqslant m|(z+1)|^{m-1} \prod_{k=1}^{l} |z^2 - 2\omega(r_k, \alpha_k)z + 1|$$
$$+ |z+1|^m \sum_{k=1}^{l} \left\{ |2z - 2\omega(r_k, \alpha_k)| \prod_{j=1; j\neq k}^{l} |z^2 - 2\omega(r_j, \alpha_j)z + 1| \right\}$$
$$\leqslant m2^{m-1} \prod_{k=1}^{l} |2 - 2\omega(r_k, \alpha_k)|$$
$$+ 2^m \sum_{k=1}^{l} \left\{ [2 + 2|\omega(r_k, \alpha_k)|] \prod_{j=1; j\neq k}^{l} |2 - 2\omega(r_j, \alpha_j)| \right\} \quad \text{(by (2.1))}$$
$$= m2^{m+l-1} \prod_{k=1}^{l} |1 - \omega(r_k, \alpha_k)|$$
$$+ 2^{m+l} \sum_{k=1}^{l} \left\{ [1 + |\omega(r_k, \alpha_k)|] \prod_{j=1; j\neq k}^{l} |1 - \omega(r_j, \alpha_j)| \right\}.$$

Hence

$$\max_{|z|=1} |p'(z)| \leqslant m2^{m+l-1} \prod_{k=1}^{l} |1 - \omega(r_k, \alpha_k)| \tag{3.6}$$
$$+ 2^{m+l} \sum_{k=1}^{l} \left\{ [1 + |\omega(r_k, \alpha_k)|] \prod_{j=1; j\neq k}^{l} |1 - \omega(r_j, \alpha_j)| \right\}.$$

From (3.4) and (3.6), it follows that

$$\frac{\max_{|z|=1}|p'(z)|}{\max_{|z|=1}|p(z)|} \leqslant \frac{m}{2} + \sum_{k=1}^{l} \frac{1 + |\omega(r_k, \alpha_k)|}{|1 - \omega(r_k, \alpha_k)|}. \tag{3.7}$$

Since $|\alpha_k| \geqslant \pi/2$, we have $\operatorname{Re}\omega(r_k, \alpha_k) \leqslant 0$ and hence by applying Lemma 2 to inequality (3.7), we have

$$\frac{\max_{|z|=1}|p'(z)|}{\max_{|z|=1}|p(z)|} \leqslant \frac{m}{2} + \sum_{k=1}^{l} 2^{1/2} \leqslant \frac{n}{2^{1/2}}$$

which gives

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{2^{1/2}} \max_{|z|=1} |p(z)|$$

and (1.3) is established.

To prove (1.4), we note that

$$\frac{p'(1)}{p(1)} = \frac{m}{2} + \sum_{k=1}^{l}\left[\frac{1}{1-r_k e^{i\alpha_k}} + \frac{1}{1-r_k^{-1}e^{-i\alpha_k}}\right]$$
$$= \frac{m}{2} + \sum_{k=1}^{l}\left[\frac{1}{1-r_k e^{i\alpha_k}} - \frac{r_k e^{i\alpha_k}}{1-r_k e^{i\alpha_k}}\right]$$
$$= m/2 + l$$
$$= n/2.$$

Therefore

(3.8) $$|p'(1)| = (n/2)|p(1)|.$$

Because $\mathrm{Max}_{|z|=1}|p(z)| = |p(1)|$, we get from (3.8),

$$\operatorname*{Max}_{|z|=1}|p'(z)| \geqslant |p'(1)| \geqslant \frac{n}{2}\operatorname*{Max}_{|z|=1}|p(z)|$$

which proves (1.4).

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