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SINGULAR REGULAR NEIGHBORHOODS AND LOCAL FLATNESS IN CODIMENSION ONE

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ABSTRACT. For an $(n-1)$-manifold S topologically embedded as a closed subset of an n-manifold N, we define what it means for S to have a singular regular neighborhood in N. The principal result demonstrates that S has a singular regular neighborhood in N if and only if the homotopy theoretic condition holds that $N - S$ is locally simply connected (1-LC) at each point of S. Consequently, S has a singular regular neighborhood in N if and only if S is locally flatly embedded $(n \neq 4)$.

In 1964 Hempel [7] obtained such results for the case $n = 3$; for a 2-manifold Σ in the 3-sphere S^3, the term he used, namely, that Σ could be deformed into each of its complementary domains, is equivalent to the property that Σ has a singular regular neighborhood in S^3. His argument for showing that $S^3 - \Sigma$ is 1-LC at each point of Σ was based on the Sphere Theorem, a result that applies only in dimension 3, and, therefore, his proof does not extend to higher dimensions. The argument given here, which provides an alternative to Hempel's for $n = 3$, employs the important result of Olum [12] that a degree one map between two orientable n-manifolds induces an epimorphism of their fundamental groups.

The notion of singular regular neighborhood has arisen in another setting: Cannon [2, Theorem 3.10] has shown that a 1-complex in a 3-manifold is tame if and only if it has a singular regular neighborhood. However, we consider no analogues of his result and deal only with singular regular neighborhoods of codimension one submanifolds. (Bryant and Lacher [16] have described an extension of this property, which they call *strong freeness*, that applies to submanifolds of arbitrary codimension.)

1. **Definitions and notation.** All manifolds are connected and boundaryless. If we refer to an $(n-1)$-manifold S in an n-manifold N, we assume S to be embedded as a closed subset of N.

Cohomology groups are computed with integer coefficients. Essentially we follow Epstein's formulation [6] of the definition of *degree of a map* (see also

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[12]); in particular we disregard negative values of degree and say that a proper map (i.e., the inverse image of each compact set is compact) $f\colon N \to M$ between orientable n-manifolds has *degree one* if f induces isomorphisms between the cohomology groups with compact supports $H_c^n(M)$ and $H_c^n(N)$.

Let S be an $(n-1)$-manifold in an n-manifold N such that $N - S$ has two components U and V. We say that S has a *singular regular neighborhood in* U if there exists a map h of $S \times I$ into $\mathrm{Cl}\ U$ such that $h(s, 0) = s$ and $h(s, t) \in U$ for all $s \in S$, $t > 0$, and that S has a *singular regular neighborhood in* N if it has singular regular neighborhoods in both U and V. Similarly, we say that S is *locally singularly collared from* U *at* $s \in S$ if there exist an open subset W of S containing s and a map f of $W \times I$ into $\mathrm{Cl}\ U$ such that $f(w, 0) = w$ and $f(w, t) \in U$ for all $w \in W$ and $t > 0$, and that S is *locally singularly bicollared at* $s \in S$ if it is locally singularly collared from both U and V at s. According to standard practice, S is said to be *locally singularly collared from* U or *locally singularly bicollared* if it has the appropriate property at each of its points.

In the case that S does not separate N, then S has a *singular regular neighborhood* if there exist a locally flat embedding g of S in an n-manifold M, a neighborhood N' of $g(S)$ in M, and a map f of N' into N such that $fg = \text{identity}$, $f(N' - g(S)) \cap S = \varnothing$, and to each $s \in S$ there correspond connected neighborhoods A_s and B_s of $g(s)$ in N' and s in N, respectively, such that $A_s - g(S)$ consists of two components A_1 and A_2 for which $f(A_1)$ and $f(A_2)$ are contained in different components of $B_s - S$. Note that in this case there is a natural and sensible definition of locally singularly bicollared and that if S has a singular regular neighborhood, then it is locally singularly bicollared in N.

2. **Degree one maps.** Throughout this section we shall suppose that Σ is an $(n-1)$-manifold in E^n and that $E^n - \Sigma$ consists of the two components U and V.

LEMMA 1. *Suppose D is an $(n-1)$-cell in Σ and $f\colon D \times I \to \mathrm{Cl}\ U$ is a map such that $f(d, 0) = d$ and $f(d, t) \in U$ for all $d \in D$, $t > 0$. Then there exists an open subset W of* $\mathrm{Cl}\ U$ *such that* $\mathrm{Int}\ D \subset W \subset f(D \times I)$.

PROOF. Let Σ' denote the closure of Σ in $S^n = E^n \cup \{\infty\}$. There exists a map H of $\Sigma' \times I$ into $\Sigma' \cup f(D \times I)$ such that

$$\begin{aligned} H(s,0) &= s && \text{for all } s \in \Sigma', \\ H(s',t) &= s' && \text{for all } s' \in \Sigma' - \mathrm{Int}\ D, \\ H(s'',t) \cap \Sigma' &= \varnothing && \text{for all } s'' \in \mathrm{Int}\ D, \quad t > 0. \end{aligned}$$

Define W as the intersection of $\mathrm{Cl}\ U$ and the component of $S^n - H(\Sigma', 1)$ that contains $\mathrm{Int}\ D$. We show that $W \subset f(D \times I)$.

Suppose to the contrary that there exists

$$w \in W - f(D \times I) \subset W - H(\Sigma' \times I).$$

Choose a point $z \in S^n - \mathrm{Cl}\ U$ and regard $S^n - \{z\}$ as E^n. As in [9, p. 97], for $x \neq z$ let π_x denote the radial map of $S^n - \{x, z\}$ onto the unit $(n-1)$-sphere centered at x. Then $\pi_w H_0$ and $\pi_w H_1$ are homotopic maps of Σ'. However, for $y \in S^n - \operatorname{Cl} U$, $y \neq z$, we see that w and y belong to different components of $S^n - H(\Sigma', 0)$ and belong to the same component of $S^n = H(\Sigma', 1)$. This leads to the desired contradiction, because according to Theorem VI.10 of [9] (and its proof), $\pi_w H_0$ is an essential map and $\pi_w H_1$ is an inessential map.

LEMMA 2. *Suppose $q \in \Sigma$ such that Σ is locally singularly collared from U at q. Then U is 1-LC at q.*

PROOF. Let $\varepsilon > 0$. The hypothesis implies that there exist (i) an $(n-1)$-cell D on Σ for which $q \in \operatorname{Int} D$ and (ii) a map f of $D \times [0,1]$ into $N_\varepsilon(q)$ such that $f(d,0) = d$ and $f(D \times (0,1]) \subset U$. Applying Lemma 1, we determine an open subset W of $\operatorname{Cl} U$ such that

$$\operatorname{Int} D \subset W \subset f(D \times I) \subset N_\varepsilon(q).$$

For the component Y of $f^{-1}(W)$ containing $\operatorname{Int} D \times \{0\}$, it follows immediately that $f|Y$ is a proper map, which implies that the degree of $f|Y$ is defined [6].

Defining $W_U = W \cap U$, $Y_U = Y \cap U$, and $f_U = f|Y_U$, we shall prove that f_U is a degree one map of Y_U to W_U. From the exact sequence

$$\cdots \to H_c^{q-1}(X - A) \to H_c^q(A) \to H_c^q(X) \to H_c^q(X - A) \to \cdots$$

of the pair (X, A), where X is locally compact and A is open in X (see [5, Chapter X]), we obtain the following commutative diagram:

$$\begin{tikzcd}
Z \arrow[d, phantom, "\wr\wr"] & Z \arrow[d, phantom, "\wr\wr"] & \\
H_c^{n-1}(\operatorname{Int} D \times \{0\}) \arrow[r, "d^*"] & H_c^n(Y_u) \arrow[r] & H_c^n(Y) \cong 0 \\
H_c^{n-1}(\operatorname{Int} D) \arrow[r, "d^*"] \arrow[u, "\cong"', "(f|\operatorname{Int} D \times \{0\})^*"] \arrow[d, phantom, "\wr\wr"] & H_c^n(W_u) \arrow[r] \arrow[u, "f_u{}^*"'] \arrow[d, phantom, "\wr\wr"] & H_c^n(W) \cong 0 \arrow[u] \\
Z & Z &
\end{tikzcd}$$

The fact that $H_c^n(Y) \cong H_c^n(W) \cong 0$ follows from [14, Theorem 6.6.10] and that the other groups are isomorphic to Z holds because the spaces are orientable manifolds. This implies that the d^*'s are isomorphisms and, therefore, that f_U is a degree one map. Consequently, f_U induces an epimorphism of $\pi_1(Y_U)$ to $\pi_1(W_U)$ [6, Corollary 3.4].

To each loop α in W_U there corresponds a loop α' in Y_U for which $(f_U)[\alpha'] = [\alpha]$. Since α' is contractible in $\operatorname{Int} D \times (0,1)$, α is contractible in $f(\operatorname{Int} D \times (0,1)) \subset N_\varepsilon(q) \cap U$. Hence, U is 1-LC at q.

REMARK. If Σ is locally collared from V at q, then we can detect geometrically that the degree of f_U is one by extending f to a map f of $D \times [-1,1]$ into $N_\varepsilon(q)$ such that f sends $D \times [-1,0]$ homeomorphically onto a collar on D from V. Because f is 1-1 on $f^{-1}f(\operatorname{Int} D \times (-1,0))$, the degree of

$$f|Y \cup (\operatorname{Int} D \times (-1,0]): Y \cup (\operatorname{Int} D \times (-1,0]) \to W \cup f(\operatorname{Int} D \times (-1,0])$$

is one [6, p. 372], and it easily follows that the degree of the restriction f_U is one also.

3. **Main results.** The following two results are immediate consequences of Lemma 2.

THEOREM 3. *Suppose S is an $(n-1)$-manifold in an n-manifold N such that $N - S$ has two components, U and V, and S is locally singularly collared from U at $s \in S$. Then U is 1-LC at s. Thus, if S has a singular regular neighborhood in U (or if S is locally singularly collared from U), U is 1-LC at each point of S.*

THEOREM 4. *Suppose S in an $(n-1)$-manifold in an n-manifold N and S is locally singularly bicollared at $s \in S$. Then $N - S$ is 1-LC at s. Thus, if S has a singular regular neighborhood in N (or if S is locally singularly bicollared), then $N - S$ is 1-LC at each point of S.*

COROLLARY 5. *Suppose S is an $(n-1)$-manifold in an n-manifold N $(n \neq 4)$ such that S has a singular regular neighborhood in N (or is locally singularly bicollared). Then S is locally flat.*

The corollary follows from [1, Theorem 7] in case $n = 3$ and from [3] or [4] in case $n \geqq 5$.

REMARK. If S separates N into two components U and V, and U is 1-LC at each point of S, it is not known $(n \geqslant 4)$ whether S is collared from U.[2] A necessary and sufficient condition $(n \geqslant 5)$ that S be collared from U is that for each $\varepsilon > 0$ there exist an ε-homeomorphism of S onto a locally flat manifold in N [13, Theorem 5].

Bryant and Lacher [16] have obtained analogues of Theorem 4 and Corollary 5 for S a manifold of arbitrary dimension embedded in N.

COROLLARY 6. *Suppose S is a locally flat $(n-1)$-manifold in an n-manifold N $(n \geqslant 4)$ and f is a map of N onto an n-manifold N' such that $f|S$ is 1-1 and $f(N - S) = N' - f(S)$. Furthermore, if $f(S)$ does not separate N', suppose that f is a proper map. Then $f(S)$ is locally flat.*

PROOF. Clearly S is locally singularly bicollared in N. A straightforward argument shows that $f(S)$ has the same property in N'.

Note that in case $f(S)$ separates N', instead of requiring that $f(N - S) = N' - f(S)$ one can require simply that $f(N - S)$ be contained in $N' - f(S)$ and meet each component of $N' - f(S)$.

EXAMPLE. To see the necessity of the hypothesis that f be proper, even in the situation where S is compact, consider a wild n-cell B in E^n such that Bd B separates the two boundary components of the annulus A given by

$$A = \{(x_1, \ldots, x_n) \in E^n | 1 \leqq x_1^2 + \cdots + x_n^2 \leqq 4\}.$$

Let $N' = S^{n-1} \times S^1$ be obtained from A by identifying the two boundary

[2] Černavskiĭ [3, Appendix 2] has announced an affirmative solution in case $n \geqslant 5$.

components of A in the natural way. Now it is a simple matter to define a map f of $E^n - B$ to N' such that, for the component S of ∂A contained in $E^n - B$, $f|S$ is 1-1, $F(S)$ equals the image of Bd B in N', and $f((E^n - B) - S) = N' - f(S)$.

For completeness we mention the following theorem. The key to the argument is that $N - S$ is k-LC at each point of S if it is 1-LC [11, Theorem 6].

THEOREM 7. *Suppose S is a $(n-1)$-manifold in an n-manifold N. Then $N - S$ is 1-LC at each point of S iff S has a singular regular neighborhood. Furthermore, in case $N - S$ has two components, U and V, U is 1-LC at each point of S iff S has a singular regular neighborhood from U.*

We close with two questions pertaining to an $(n-1)$-manifold S in E^n and a component U of $E^n - S$. The second question is unsolved even for $n = 3$, while the first has an affirmative solution in that case [15, Theorem 6].

Question 1. Suppose for each open subset W of S and each continuous function $\varepsilon\colon W \to (0, \infty)$ there exists a map $f\colon W \to U$ such that $\rho(w, f(w)) < \varepsilon(w)$ for each $w \in W$. Is U 1-LC at each point of S?

Question 2 (Free Surface Problem). If for each map $\varepsilon\colon S \to (0, \infty)$ there exists a map $f\colon S \to U$ such that $\rho(s, f(s)) < \varepsilon(s)$ for all $s \in S$, is C 1-LC at each point of S?

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