PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 57, Number 2, June 1976

A NOTE ON THE CONTINUITY OF LOCAL TIMES[1]

DONALD GEMAN

ABSTRACT. Several conditions are given for a stochastic process $X(t)$ on $[0, 1]$ to have a local time which is continuous in its time parameter (for example, in the Gaussian case, the integrability of $[E(X(t) - X(s))^2]^{-1/2}$ over the unit square). Furthermore, for any Borel function F on $[0, 1]$ with a continuous local time, the approximate limit of $|F(s) - F(t)|/|s - t|$ as $s \to t$ is infinite for a.e. $t \in [0, 1]$ and $\{s | F(s) = F(t)\}$ is uncountable for a.e. $t \in [0, 1]$.

1. For a real, Borel function $F(t)$, $0 \leqslant t \leqslant 1$, put

$$\mu_t(B) = m(F^{-1}(B) \cap [0, t]), \quad B \in \mathfrak{B},$$

where $m(dx)$ (or just dx) is Lebesgue measure on the real Borel σ-field $\mathfrak{B}$. If μ_1 is absolutely continuous with respect to m, we call $\alpha_t(x) \equiv d\mu_t(x)/dm$, $0 \leqslant t \leqslant 1$, $-\infty < x < \infty$, the *local time* of F. Local times are assumed to be chosen jointly measurable and nondecreasing, right-continuous in t for every x; such versions always exist. The measure corresponding to $\alpha_.(x)$ is then denoted $\alpha(dt, x)$ and represents the "time spent" by F in the level x during dt.

Let $X(t, w)$, $0 \leqslant t \leqslant 1$, be a separable and measurable stochastic process over a probability space $(\Omega, \mathfrak{F}, P)$. In the Gaussian case, mean 0, Berman [1] showed that if

$$\int_0^1 \int_0^1 \frac{dsdt}{[E(X(s) - X(t))^2]^{1/2}} < \infty, \tag{1}$$

then almost every sample path $X(\cdot, w)$ has a local time, say $\alpha_t(x, w)$, and then, in a series of papers, gave various additional conditions for a version of $\alpha_t(x, w)$ to be jointly continuous in (t, x) a.s. (for example, in the stationary increments case, (1) with the exponent 1/2 replaced by $1 + \varepsilon$ for some $\varepsilon > 0$; see [3]). Berman [3] also observed that for any Borel function F, the joint continuity of its local time implies

$$\operatorname{ap}\lim_{s \to t} \left| \frac{F(s) - F(t)}{s - t} \right| = +\infty \tag{2}$$

for *every* $t \in (0, 1)$ (where "ap lim" stands for approximate limit–see [6, p.

Received by the editors July 11, 1975.
AMS (MOS) subject classifications (1970). Primary 60G15, 60G17, 26A24.
[1] This work was partially supported by NSF Grant MPS72-04813 A03.

220] and below), and in [2] proved that if (1) holds, and $X(\cdot, w)$ is continuous, then, with probability one, $\{s \in [0,1] | X(s,w) = X(t,w)\}$ is infinite for a.e. $t \in [0,1]$.

In this note we will prove:

THEOREM A. *Suppose F has a local time $\alpha_t(x)$ which is continuous in t for a.e. x. Then*

(a) ap $\lim_{s \to t} |(F(s) - F(t))/(s-t)| = +\infty$ *for a.e.* $t \in [0,1]$,

(b) $L_t = \{s \in [0,1] | F(s) = F(t)\}$ *is uncountable for a.e.* $t \in [0,1]$.

THEOREM B. *Let $X(t,w)$ be a stochastic process for which either*

$$\text{(I)} \qquad \int_0^1 \sup_{\varepsilon > 0} \frac{1}{\varepsilon} P(|X(s) - X(t)| \leqslant \varepsilon) ds < \infty \quad \textit{for a.e. } t \in [0,1],$$

or

(II) *For each $0 \leqslant s \leqslant 1$, the distribution of $X(s)$ is absolutely continuous; for each $0 \leqslant s, t \leqslant 1$, the (joint) distribution of $(X(s), X(t))$ is absolutely continuous on an open strip B (independent of s, t) containing the diagonal $x = y$, and the density $g_{s,t}(x,y)$ on B is continuous on the diagonal and satisfies*

$$\text{(3)} \qquad \int_0^1 \int_0^1 \sup_{x,y \in B} g_{s,t}(x,y) ds dt < \infty.$$

Then, with probability one, $X(t,w)$ has a local time $\alpha_t(x,w)$ which is continuous in t for a.e. x.

In the Gaussian case, (I) becomes

$$\text{(4)} \qquad \int_0^1 \frac{ds}{[E(X(t) - X(s))^2]^{1/2}} < \infty \quad \text{for a.e. } t \in [0,1],$$

which is implied by (1). (However, the conclusions inferred from (1) can also be inferred from (4) by restricting $X(t,w)$ to a sequence $E_n \subset [0,1]$ with

$$m\Big([0,1] \setminus \bigcup_n E_n\Big) = 0$$

and $[E(X(t) - X(s))^2]^{-1/2}$ integrable over each $E_n \times E_n$.) Condition (4) (or (1)), is weaker than the known sufficient conditions for the *joint* continuity of α. Still in the Gaussian case (mean 0), (II) reduces to

$$\text{(5)} \qquad \int_0^1 \int_0^1 \frac{ds dt}{[(EX^2(t))(EX^2(s)) - (EX(t)X(s))^2]^{1/2}} < \infty,$$

which was assumed in [5] for the existence of α. For a continuous covariance, (5) implies (1), but not conversely, as seen by the process $W(t^2, w)$, W being ordinary Brownian motion. Nonetheless, if $X(t,w)$ satisfies (1), then the process $X(t,w) + \xi(w)$, where ξ is standard normal and independent of X, has the same local time properties as X and satisfies *both* (1) and (5). In the stationary case, (1), (4), and (5) are equivalent, and are implied by

$$\int_0^1 \frac{ds}{\sigma(s)} < \infty, \qquad \sigma^2(s) = E[X(s) - E(X(s)\,|\,X(u), u \leqslant 0)]^2,$$

which was assumed in [4] to get the continuity (in t) of α.

The meaning of part (a) of Theorem A is this: almost every $t \in (0, 1)$ is a point of dispersion for the time spent by F in every cone with vertex at $(t, F(t))$ and axis parallel to the abscissa. In particular, almost every $t \in (0, 1)$ is an approximate "knot-point" (a term due to G. C. Young) of F, i.e. both approximate upper Dini derivates are $+\infty$ and both lower ones are $-\infty$ (and hence likewise for the ordinary derivates). (This follows from the Denjoy-Khintchine theorem [6, p. 295].) It is noteworthy that for any Borel function F, the set of points t at which $\lim_{s\to t}|F(s) - F(t)|/|t - s| = +\infty$ is of measure zero [6, p. 270]. This of course is false if lim is replaced by ap lim: "this rather unexpected fact was brought to light by V. Jarnik, who showed that there exist continuous functions F for which the relation

$$\operatorname{ap}\lim_{h\to 0^+} |F(x + h) - F(x)|/h = +\infty$$

holds at almost all points x" [6, p. 297].

2. For brevity, the verification of the measurability of various sets and functions will be left aside.

PROOF OF THEOREM A. (a) For each $0 < t < 1$ and $M > 0$, set

$$D_{t,M} = \{s \in [0, 1] \,|\, |F(s) - F(t)| \leqslant M|t - s|\}, \quad \text{and}$$

$$\tau(t, M) = \overline{\lim_{\varepsilon\downarrow 0}}\, \frac{1}{2\varepsilon} m(D_{t,M} \cap (t - \varepsilon, t + \varepsilon)).$$

By definition, (2) holds at t whenever $\tau(t, M) = 0$ for every $M > 0$.

It follows from the definition of α and a monotone class argument that

$$\int_0^1 f(s, F(s))\,ds = \int_{-\infty}^{\infty} \int_0^1 f(s, x)\alpha(ds, x)\,dx \tag{6}$$

for every measurable $f \geqslant 0$. In particular, for every $0 \leqslant s < t \leqslant 1$ there is a full set $E_{s,t}$, i.e. $m(E_{s,t}^c) = 0$, such that

$$\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}(\mu_t[x - \varepsilon, x + \varepsilon] - \mu_s[x - \varepsilon, x + \varepsilon]) = \alpha((s, t), x) \quad \text{for all } x \in E_{s,t}.$$

Let $\mathcal{K}$ denote the collection of open intervals in $[0, 1]$ with rational endpoints. Since $\mathcal{K}$ is countable,

$$\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_B I_{[0,\varepsilon]}(|F(s) - x|)\,ds = \alpha(B, x) \quad \forall B \in \mathcal{K}, \tag{7}$$

for all x in a full set E. Now (7) remains valid with x replaced by $F(t)$ for all t in $G \equiv F^{-1}(E)$, which is full since $m(E^c) = 0$ implies $m(F^{-1}(E^c)) = 0$.

Now fix $B \in \mathcal{K}$, $t \in B \cap G$, and $M > 0$.

$$\alpha(B, F(t)) \geqslant \varlimsup_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_{B\cap D_{t,M}} I_{[0,\varepsilon]}(|F(s) - F(t)|)\, ds$$

$$\geqslant \varlimsup_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_{B\cap D_{t,M}} I_{[0,\varepsilon]}(M|t - s|)\, ds$$

$$= \varlimsup_{\varepsilon\downarrow 0} \frac{1}{M}\left[\frac{1}{2\varepsilon} m(B \cap D_{t,M} \cap (t - \varepsilon, t + \varepsilon))\right]$$

$$= \tau(t, M)/M.$$

Letting $B \downarrow \{t\}$, we find that

$$\sup_{M>0} \frac{\tau(t, M)}{M} \leqslant \alpha(\{t\}, F(t)), \quad t \in G, \tag{8}$$

where $\alpha(\{t\}, x)$ is the mass placed on $\{t\}$ by $\alpha(ds, x)$.

Finally, let $D = \{x\colon \alpha(\{s\}, x) = 0\ \forall s\}$: by assumption, $m(D^c) = 0$, and hence $m(F^{-1}(D^c)) = 0$, i.e. $\alpha(\{s\}, F(t)) = 0\ \forall s$ for a.e. $t \in [0, 1]$. In particular, $\alpha(\{t\}, F(t)) = 0$ a.e.

(b) Having already noticed that the measure $\alpha(ds, F(t))$ is continuous for a.e. t, we need only check that (i) $\alpha_1(F(t)) > 0$ a.e. and (ii) $\alpha(L_t^c, F(t)) = 0$ a.e.

As for (i), for any $B \in \mathfrak{B}$, $B \subset [0, 1]$,

$$\int_B \alpha_1(F(s))ds = \int_{-\infty}^{\infty} \alpha_1(x)\alpha(B, x)dx \geqslant \int_{-\infty}^{\infty} \alpha^2(B, x)dx,$$

which is positive whenever $m(B) > 0$. (The assumption in [2, Lemma 1.1] that α be square integrable is extraneous.) To obtain (ii), let $M_x = \{s \in [0, 1] | F(s) = x\}$ so that $L_t = M_{F(t)}$. From (6),

$$0 = \int_0^1 I_{L_t^c}(t)\, dt = \int_{-\infty}^{\infty} \alpha(M_x^c, x)\, dx,$$

in which case $\alpha(M_x^c, x) = 0$ a.e., and in turn $\alpha(L_t^c, F(t)) = 0$ for a.e. $t \in [0, 1]$.

Proof of Theorem B. Assumption (I) implies

$$\varliminf_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^1 I_{[0,\varepsilon]}(|X(s) - X(t)|)\, ds < \infty \quad \text{for a.e.} \quad t \in [0, 1], \tag{9}$$

for almost every $w \in \Omega$ (by taking the expected value of the random variable in (9) and using Fatou's lemma). However, for *any* Borel function $F(t)$ on $[0, 1]$, (9)–with X replaced by F–is necessary *and* sufficient for a local time. To see this, let $G(x) = m(F^{-1}(-\infty, x] \cap [0, 1])$; it is a standard fact about the differentiation of measures that $G'(x)$ exists, finite *or* infinite, for a.e. $x(dG)$, and that G is absolutely continuous if and only if $G'(x) < \infty$ a.e. (dG). In other words,

$$\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^1 I_{[0,\varepsilon]}(|F(s) - F(t)|)\, ds$$

exists (possibly at $+\infty$) for a.e. $t \in [0, 1]$, and F has a local time if and only if the limit is actually finite for a.e. $t \in [0,1]$.

Next, proceeding as in the proof of Theorem A, we have

$$\alpha(B, X(t,w), w) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_B I_{[0,\varepsilon]}(|X(s,w) - X(t,w)|)\, ds, \quad \forall B \in \mathcal{K},$$

for a.e. $t \in [0,1]$, all with probability one. Taking expected values and applying Fatou's lemma yields

$$E\alpha(B, X(t)) \leqslant \int_B \sup_{\varepsilon > 0} \frac{1}{\varepsilon} P(|X(t) - X(s)| \leqslant \varepsilon)\, ds, \quad \forall B \in \mathcal{K},$$

for a.e. $t \in [0,1]$, and for such t's we can choose a sequence from $\mathcal{K}$ which decreases to $\{t\}$ and conclude that $E\alpha(\{t\}, X(t,w), w) = 0$. Consequently with probability one,

$$\begin{aligned} 0 &= \int_0^1 \alpha(\{t\}, X(t,w), w)\, dt = \int_{-\infty}^{\infty} \int_0^1 \alpha(\{t\}, x, w)\alpha(dt, x, w)\, dx \\ &= \int_{-\infty}^{\infty} \sum_{t \in [0,1]} \alpha^2(\{t\}, x, w)\, dx, \end{aligned}$$

where the second equality uses (6).

Under condition (II), the existence of α results from an easy modification of the proof of Theorem 2 of [5]. As for continuity, let $\phi(\lambda; x, w)$ be the Fourier transform of the (finite) measure $\alpha(dt, x, w)$:

$$\phi(\lambda; x, w) \triangleq \int e^{i\lambda t} \alpha(dt, x, w), \qquad -\infty < \lambda, x < +\infty, \quad w \in \Omega.$$

As is well known,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |\phi(\lambda; x, w)|^2 d\lambda = \sum_{t \in [0,1]} \alpha^2(\{t\}, x, w). \tag{10}$$

We will show that the expected value of the left-hand side of (10) is zero for a.e. x.

From (6), for almost every w,

$$\int_0^1 e^{i\lambda s} I_{[0,\varepsilon]}(X(s,w) - x) ds = \int_x^{x+\varepsilon} \phi(\lambda; y, w) dy \quad \forall \lambda, x. \tag{11}$$

Let $\varepsilon Z_\varepsilon(\lambda; x, w)$ denote the left-hand side of (11). Almost surely, then, for every λ,

$$\lim_{\varepsilon \downarrow 0} Z_\varepsilon(\lambda; x, w) = \phi(\lambda; x, w) \tag{12}$$

for a.e. x. Using Fubini's theorem we obtain set $\Delta \in \mathcal{B}$, $m(\Delta^c) = 0$, such that for every $x \in \Delta$, (12) holds for $(m \times P)$-a.e. pair (λ, w). For such x's,

$$
\begin{aligned}
E \lim_{T\to\infty} \frac{1}{T}\int_0^T |\phi(\lambda; x, w)|^2\, d\lambda &\leqslant \varliminf_{T\to\infty} E\frac{1}{T}\int_0^T |\phi(\lambda; x, w)|^2\, d\lambda \\
&= \varliminf_{T\to\infty} \frac{1}{T}\int_0^T E|\phi(\lambda; x, w)|^2\, d\lambda \\
&= \varliminf_{T\to\infty} \frac{1}{T}\int_0^T E \lim_{\varepsilon\downarrow 0} |Z_\varepsilon(\lambda; x, w)|^2\, d\lambda \\
&\leqslant \varliminf_{T\to\infty} \frac{1}{T}\int_0^T \varliminf_{\varepsilon\downarrow 0} E|Z_\varepsilon(\lambda; x, w)|^2\, d\lambda.
\end{aligned}
$$

Finally, by dominated convergence and the continuity of $g_{s,t}$ at (x, x),

$$
\begin{aligned}
\varliminf_{\varepsilon\downarrow 0} E|Z_\varepsilon(\lambda; x, w)|^2 &= \varliminf_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^2}\int_0^1\int_0^1\int_x^{x+\varepsilon}\int_x^{x+\varepsilon} e^{i\lambda(s-t)} g_{s,t}(u, v)\, du\, dv\, ds\, dt \\
&= \int_0^1\int_0^1 e^{i\lambda(s-t)} g_{s,t}(x, x)\, ds\, dt \\
&= \int_0^1 e^{i\lambda s} G(s; x)\, ds,
\end{aligned}
$$

where $G(s; x) = \int_0^1 I_{[0,1]}(s + t) g_{t,s+t}(x, x)\, dt$ is nonnegative and integrable (ds) over $(-\infty, \infty)$. Hence

$$
\lim_{T\to\infty} \frac{1}{T}\int_0^T\int_{-\infty}^{\infty} e^{i\lambda s} G(s; x)\, ds d\lambda = 0 \quad \text{for every } x,
$$

which concludes the proof.

REMARK. For a Gaussian process satisfying (1), we have $E\alpha(1, x) > 0$ for a.e. x. Consequently, the proof of part (b) shows that, for almost every x, $M_x(w) = \{t \in [0, 1] | X(t, w) = x\}$ is uncountable with positive probability. Under further restrictions, Berman [3], Orey [5], and others have computed the a.s. Hausdorff dimension of $M_x(w)$.

REFERENCES

1. S. M. Berman, *Local times and sample function properties of stationary Gaussian processes*, Trans. Amer. Math. Soc. **137** (1969), 277–299. MR **39** #1009.
2. ———, *Harmonic analysis of local times and sample functions of Gaussian processes*, Trans. Amer. Math. Soc. **143** (1969), 269–281. MR **40** #2155.
3. ———, *Gaussian processes with stationary increments: Local times and sample function properties*, Ann. Math. Statist. **41** (1970), 1260–1272. MR **42** #6916.
4. D. Geman and J. Horowitz, *Local times and supermartingales*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **29** (1974), 273–293. MR **50** #11476.
5. S. Orey, *Gaussian sample functions and the Hausdorff dimension of level crossings*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **17** (1971), 39–47.
6. S. Saks, *Theory of the integral*, Dover, New York, 1964. MR **29** #4850.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MASSACHUSETTS, AMHERST, MASSACHUSETTS 01002