A NOTE ON THE CONTINUITY OF LOCAL TIMES

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Abstract. Several conditions are given for a stochastic process \( X(t) \) on \([0, 1]\) to have a local time which is continuous in its time parameter (for example, in the Gaussian case, the integrability of \( \int_0^1 \left( \mathbb{E}(X(t) - X(s))^2 \right)^{1/2} \, ds \) over the unit square). Furthermore, for any Borel function \( F \) on \([0, 1]\) with a continuous local time, the approximate limit of \( \frac{|F(s) - F(t)|}{|s - t|} \) as \( s \to t \) is infinite for a.e. \( t \in [0, 1] \) and \( \{ s | F(s) = F(t) \} \) is uncountable for a.e. \( t \in [0, 1] \).

1. For a real, Borel function \( F(t) \), \( 0 < t < 1 \), put
\[
m_t(B) = m(F^{-1}(B) \cap [0, t]), \quad B \in \mathcal{B},
\]
where \( m(dx) \) (or just \( dx \)) is Lebesgue measure on the real Borel \( \sigma \)-field \( \mathcal{B} \). If \( m_t \) is absolutely continuous with respect to \( m \), we call \( \alpha_t(x) = \frac{d m_t(x)}{dm} \), \( 0 < t < 1, -\infty < x < \infty \), the local time of \( F \). Local times are assumed to be chosen jointly measurable and nondecreasing, right-continuous in \( t \) for every \( x \); such versions always exist. The measure corresponding to \( \alpha_t(x) \) is then denoted \( \alpha(dt, x) \) and represents the “time spent” by \( F \) in the level \( x \) during \( dt \).

Let \( X(t, w), 0 \leq t \leq 1, \) be a separable and measurable stochastic process over a probability space \((\Omega, \mathcal{F}, P)\). In the Gaussian case, mean 0, Berman [1] showed that if
\[
\int_0^1 \int_0^1 \frac{dsdt}{[E(X(s) - X(t))^2]^{1/2}} < \infty,
\]
then almost every sample path \( X(\cdot, w) \) has a local time, say \( \alpha_t(x, w) \), and then, in a series of papers, gave various additional conditions for a version of \( \alpha_t(x, w) \) to be jointly continuous in \((t, x)\) a.s. (for example, in the stationary increments case, (1) with the exponent 1/2 replaced by \( 1 + \epsilon \) for some \( \epsilon > 0 \); see [3]). Berman [3] also observed that for any Borel function \( F \), the joint continuity of its local time implies
\[
\text{ap lim}_{s \to t} \left| \frac{F(s) - F(t)}{s - t} \right| = +\infty
\]
for every \( t \in (0, 1) \) (where “ap lim” stands for approximate limit—see [6, p.

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and in [2] proved that if (1) holds, and $X(t, w)$ is continuous, then, with probability one, \( \{ s \in [0, 1] \mid X(s, w) = X(t, w) \} \) is infinite for a.e. $t \in [0, 1]$.

In this note we will prove:

**Theorem A.** Suppose $F$ has a local time $\alpha_t(x)$ which is continuous in $t$ for a.e. $x$. Then
\begin{enumerate}[(a)]  
  
  \item \( \text{ap lim}_{s \to t} \frac{|F(s) - F(t)|}{s - t} = +\infty \) for a.e. $t \in [0, 1]$,  
  
  \item $L_t = \{ s \in [0, 1] \mid F(s) = F(t) \}$ is uncountable for a.e. $t \in [0, 1]$.  
\end{enumerate}

**Theorem B.** Let $X(t, w)$ be a stochastic process for which either
\begin{equation}
(1) \quad \int_0^1 \sup_{\epsilon > 0} \frac{1}{\epsilon} P(|X(s) - X(t)| < \epsilon) \, ds < \infty \quad \text{for a.e. } t \in [0, 1],
\end{equation}
or
\begin{equation}
(II) \quad \text{For each } 0 < s < 1, \text{ the distribution of } X(s) \text{ is absolutely continuous; for each } 0 < s, t < 1, \text{ the (joint) distribution of } (X(s), X(t)) \text{ is absolutely continuous on an open strip } B \text{ (independent of } s, t \text{) containing the diagonal } x = y, \text{ and the density } g_{s,t}(x,y) \text{ on } B \text{ is continuous on the diagonal and satisfies}
\end{equation}
\begin{equation}
(3) \quad \int_0^1 \int_0^1 \sup_{x,y \in B} g_{s,t}(x,y) \, ds \, dt < \infty.
\end{equation}

Then, with probability one, $X(t, w)$ has a local time $\alpha_t(x, w)$ which is continuous in $t$ for a.e. $x$.

In the Gaussian case, (I) becomes
\begin{equation}
(4) \quad \int_0^1 \frac{ds}{[E(X(t) - X(s))^2]^{1/2}} < \infty \quad \text{for a.e. } t \in [0, 1],
\end{equation}
which is implied by (1). (However, the conclusions inferred from (1) can also be inferred from (4) by restricting $X(t, w)$ to a sequence $E_n \subset [0, 1]$ with
\begin{equation}
m([0, 1] \setminus \bigcup_n E_n) = 0
\end{equation}
and $[E(X(t) - X(s))^2]^{-1/2}$ integrable over each $E_n \times E_n$.) Condition (4) (or (1)), is weaker than the known sufficient conditions for the joint continuity of $\alpha$. Still in the Gaussian case (mean 0), (II) reduces to
\begin{equation}
(5) \quad \int_0^1 \int_0^1 \frac{ds \, dt}{[(EX^2(t))(EX^2(s)) - (EX(t)X(s))^2]^{1/2}} < \infty,
\end{equation}
which was assumed in [5] for the existence of $\alpha$. For a continuous covariance, (5) implies (1), but not conversely, as seen by the process $W(t^2, w)$, $W$ being ordinary Brownian motion. Nonetheless, if $X(t, w)$ satisfies (1), then the process $X(t, w) + \xi(w)$, where $\xi$ is standard normal and independent of $X$, has the same local time properties as $X$ and satisfies both (1) and (5). In the stationary case, (1), (4), and (5) are equivalent, and are implied by
\[ \int_0^1 \frac{ds}{\sigma(s)} < \infty, \quad \sigma^2(s) = E[X(s) - E(X(s)|X(u), u \leq 0)]^2, \]

which was assumed in [4] to get the continuity (in \( t \)) of \( \alpha \).

The meaning of part (a) of Theorem A is this: almost every \( t \in (0, 1) \) is a point of dispersion for the time spent by \( F \) in every cone with vertex at \((t, F(t))\) and axis parallel to the abscissa. In particular, almost every \( t \in (0, 1) \) is an approximate "knot-point" (a term due to G. C. Young) of \( F \), i.e. both approximate upper Dini derivatives are \(+\infty\) and both lower ones are \(-\infty\) (and hence likewise for the ordinary derivatives). (This follows from the Denjoy-Khintchine theorem [6, p. 295].) It is noteworthy that for any Borel function \( F \), the set of points \( t \) at which \( \lim_{s \to t} |F(s) - F(t)|/|t - s| = +\infty \) is of measure zero [6, p. 270]. This of course is false if \( \lim \) is replaced by ap \( \lim \): "this rather unexpected fact was brought to light by V. Jarnik, who showed that there exist continuous functions \( F \) for which the relation

\[ \text{ap \( \lim_{h \to 0^+} \frac{|F(x + h) - F(x)|}{h} = +\infty \) holds at almost all points \( x \)" [6, p. 297].

2. For brevity, the verification of the measurability of various sets and functions will be left aside.

PROOF OF THEOREM A. (a) For each \( 0 < t < 1 \) and \( M > 0 \), set

\[ D_{t,M} = \{ s \in [0, 1] | |F(s) - F(t)| \leq M|t - s| \}, \quad \text{and} \]

\[ \tau(t, M) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} m(D_{t,M} \cap (t - \epsilon, t + \epsilon)). \]

By definition, (2) holds at \( t \) whenever \( \tau(t, M) = 0 \) for every \( M > 0 \).

It follows from the definition of \( \alpha \) and a monotone class argument that

\[ \int_0^1 f(s, F(s)) ds = \int_{-\infty}^\infty \int_0^1 f(s, x) \alpha(ds, x) dx \]

for every measurable \( f \geq 0 \). In particular, for every \( 0 \leq s < t \leq 1 \) there is a full set \( E_{s,t} \), i.e. \( m(E_{s,t}^C) = 0 \), such that

\[ \lim_{\epsilon \to 0} \frac{1}{2\epsilon} (\mu_{t}[x - \epsilon, x + \epsilon] - \mu_{\epsilon}[x - \epsilon, x + \epsilon]) = \alpha((s,t),x) \quad \text{for all} \ x \in E_{s,t}. \]

Let \( \mathcal{K} \) denote the collection of open intervals in \([0,1]\) with rational endpoints. Since \( \mathcal{K} \) is countable,

\[ \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_B \mathbb{1}_{[0,\epsilon]}(|F(s) - x|) ds = \alpha(B, x) \quad \forall B \in \mathcal{K}, \]

for all \( x \) in a full set \( E \). Now (7) remains valid with \( x \) replaced by \( F(t) \) for all \( t \in G \equiv F^{-1}(E^c) \), which is full since \( m(E^c) = 0 \) implies \( m(F^{-1}(E^c)) = 0 \).

Now fix \( B \in \mathcal{K}, \ t \in B \cap G, \) and \( M > 0 \).
\[\alpha(B, F(t)) \geq \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{B \cap D_{t,M}} I_{[0, \epsilon]}(|F(s) - F(t)|) \, ds\]

\[\geq \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{B \cap D_{t,M}} I_{[0, \epsilon]}(M|t - s|) \, ds\]

\[= \lim_{\epsilon \to 0} \frac{1}{M} \left[ \frac{1}{2\epsilon} m(B \cap D_{t,M} \cap (t - \epsilon, t + \epsilon)) \right]\]

\[= \tau(t, M)/M.\]

Letting \( B \downarrow \{t\} \), we find that

\[\sup_{M > 0} \frac{\tau(t, M)}{M} \leq \alpha({t}, F(t)), \quad t \in G,\]

where \( \alpha({t}, x) \) is the mass placed on \( \{t\} \) by \( \alpha(ds, x) \).

Finally, let \( D = (x: \alpha([s], x) = 0 \, \forall s) \): by assumption, \( m(D^c) = 0 \), and hence \( m(F^{-1}(D^c)) = 0 \), i.e. \( \alpha([s], F(t)) = 0 \, \forall s \) for a.e. \( t \in [0, 1] \). In particular, \( \alpha([t], F(t)) = 0 \) a.e.

(b) Having already noticed that the measure \( \alpha(ds, F(t)) \) is continuous for a.e. \( t \), we need only check that (i) \( \alpha_1(F(t)) > 0 \) a.e. and (ii) \( \alpha(L_{t},F(t)) = 0 \) a.e.

As for (i), for any \( B \in \mathfrak{B}, B \subset [0, 1], \)

\[\int_B \alpha_1(F(s)) \, ds = \int_{-\infty}^{\infty} \alpha_1(x) \alpha(B, x) \, dx \geq \int_{-\infty}^{\infty} a^2(B, x) \, dx,\]

which is positive whenever \( m(B) > 0 \). (The assumption in [2, Lemma 1.1] that \( \alpha \) be square integrable is extraneous.) To obtain (ii), let \( M_x = \{s \in [0, 1] | F(s) = x\} \) so that \( L_t = M_{F(t)} \). From (6),

\[0 = \int_0^1 L_{t}^1 (t) \, dt = \int_{-\infty}^{\infty} \alpha(M_x^c, x) \, dx,\]

in which case \( \alpha(M_x^c, x) = 0 \) a.e., and in turn \( \alpha(L_{t},F(t)) = 0 \) for a.e. \( t \in [0, 1] \).

**Proof of Theorem B.** Assumption (I) implies

\[\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^1 I_{[0, \epsilon]}(|X(s) - X(t)|) \, ds < \infty \quad \text{for a.e.} \quad t \in [0, 1],\]

for almost every \( w \in \Omega \) (by taking the expected value of the random variable in (9) and using Fatou's lemma). However, for any Borel function \( F(t) \) on \([0, 1], (9)-with X replaced by F-is necessary and sufficient for a local time. To see this, let \( G(x) = m(F^{-1}(-\infty, x] \cap [0, 1]) \); it is a standard fact about the differentiation of measures that \( G'(x) \) exists, finite or infinite, for a.e. \( x(dG) \), and that \( G \) is absolutely continuous if and only if \( G'(x) < \infty \) a.e. (dG). In other words,

\[\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^1 I_{[0, \epsilon]}(|F(s) - F(t)|) \, ds \]

exists (possibly at \( +\infty \)) for a.e. \( t \in [0, 1] \), and \( F \) has a local time if and only if
the limit is actually finite for a.e. \( t \in [0,1] \).

Next, proceeding as in the proof of Theorem A, we have

\[
\alpha(B, X(t, w), w) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_B \int_{[0,\varepsilon]} |X(s, w) - X(t, w)| \, ds, \quad \forall B \in \mathcal{H},
\]

for a.e. \( t \in [0,1] \), all with probability one. Taking expected values and applying Fatou's lemma yields

\[
E\alpha(B, X(t)) \leq \int_B \sup_{\varepsilon > 0} \frac{1}{\varepsilon} P(|X(t) - X(s)| \leq \varepsilon) \, ds, \quad \forall B \in \mathcal{H},
\]

for a.e. \( t \in [0,1] \), and for such \( t \)'s we can choose a sequence from \( \mathcal{H} \) which decreases to \( \{t\} \) and conclude that \( E\alpha(\{t\}, X(t, w), w) = 0 \). Consequently with probability one,

\[
0 = \int_0^1 \alpha(\{t\}, X(t, w), w) \, dt = \int_{-\infty}^\infty \int_0^1 \alpha(\{t\}, x, w) \alpha(dt, x, w) \, dx
\]

where the second equality uses \( (6) \).

Under condition (II), the existence of \( \alpha \) results from an easy modification of the proof of Theorem 2 of [5]. As for continuity, let \( \phi(\lambda; x, w) \) be the Fourier transform of the (finite) measure \( \alpha(dt, x, w) \):

\[
\phi(\lambda; x, w) \triangleq \int e^{i\lambda s} \alpha(dt, x, w), \quad -\infty < \lambda, x < +\infty, \ w \in \Omega.
\]

As is well known,

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T |\phi(\lambda; x, w)|^2 \, d\lambda = \sum_{t \in [0,1]} \alpha^2(\{t\}, x, w).
\]

We will show that the expected value of the left-hand side of \( (10) \) is zero for a.e. \( x \).

From \( (6) \), for almost every \( w \),

\[
\int_0^1 e^{i\lambda s} \int_{[0,\varepsilon]} (X(s, w) - x) \, ds = \int_x^{x+\varepsilon} \phi(\lambda; y, w) \, dy, \quad \forall \lambda, x.
\]

Let \( eZ_\varepsilon(\lambda; x, w) \) denote the left-hand side of \( (11) \). Almost surely, then, for every \( \lambda \),

\[
\lim_{\varepsilon \downarrow 0} Z_\varepsilon(\lambda; x, w) = \phi(\lambda; x, w)
\]

for a.e. \( x \). Using Fubini's theorem we obtain set \( \Delta \in \mathcal{B}, m(\Delta^c) = 0 \), such that for every \( x \in \Delta \), \( (12) \) holds for \( (m \times P) \)-a.e. pair \( (\lambda, w) \). For such \( x \)'s,
\[
E \lim_{T \to \infty} \frac{1}{T} \int_0^T |\phi(\lambda; x, w)|^2 d\lambda \leq \lim_{T \to \infty} \frac{1}{T} \int_0^T |\phi(\lambda; x, w)|^2 d\lambda
\]
\[
= \lim_{T \to \infty} \frac{1}{T} \int_0^T E|\phi(\lambda; x, w)|^2 d\lambda
\]
\[
= \lim_{T \to \infty} \frac{1}{T} \int_0^T E \lim_{\epsilon \to 0} |Z_\epsilon(\lambda; x, w)|^2 d\lambda
\]
\[
\leq \lim_{T \to \infty} \frac{1}{T} \int_0^T \lim_{\epsilon \to 0} E|Z_\epsilon(\lambda; x, w)|^2 d\lambda.
\]

Finally, by dominated convergence and the continuity of \( g_{s,t} \) at \((x, x)\),
\[
\lim_{\epsilon \to 0} E|Z_\epsilon(\lambda; x, w)|^2 = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \int_0^1 \int_0^1 \int_x^{x+\epsilon} \int_x^{x+\epsilon} e^{i\lambda(s-t)} g_{s,t}(u, v) du dv ds dt
\]
\[
= \int_0^1 \int_0^1 e^{i\lambda(s-t)} g_{s,t}(x, x) ds dt
\]
\[
= \int_0^1 e^{i\lambda s} G(s; x) ds,
\]
where \( G(s; x) = \int_0^1 I_{[0,1]}(s+t) g_{s,t}(x, x) dt \) is nonnegative and integrable \((ds)\) over \((-\infty, \infty)\). Hence
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \int_0^\infty e^{i\lambda s} G(s; x) ds d\lambda = 0 \quad \text{for every } x,
\]
which concludes the proof.

**Remark.** For a Gaussian process satisfying (1), we have \( E\alpha(1, x) > 0 \) for a.e. \( x \). Consequently, the proof of part (b) shows that, for almost every \( x \), \( M_x(w) = \{t \in [0,1] | X(t, w) = x) \) is uncountable with positive probability. Under further restrictions, Berman [3], Orey [5], and others have computed the a.s. Hausdorff dimension of \( M_x(w) \).

**References**