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CONVERSES TO MEASURABILITY THEOREMS FOR YEH-WIENER SPACE

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ABSTRACT. In this paper we establish some relationships between Yeh-Wiener measurability and Wiener measurability of certain sets and functionals. In addition we show that an "interval" in Yeh-Wiener space is Yeh-Wiener measurable if and only if its "restriction set" in Euclidean space is Lebesgue measurable.

0. **Introduction.** Let $C_1[a,b]$ denote the Wiener space of one variable, i.e. $C_1[a,b] = \{x(\cdot) \mid x(a) = 0 \text{ and } x(s) \text{ is continuous on } [a,b]\}$. Let $R = \{(s,t) \mid a \leqslant s \leqslant b,\ a < t < \beta\}$ and let $C_2[R]$, called Yeh-Wiener space, denote the Wiener space of functions of two variables over R, i.e. $C_2[R] = \{x(\cdot,\cdot) \mid x(a,t) = x(s,\alpha) = 0 \text{ and } x(s,t) \text{ is continuous on } R\}$.

In [2] Cameron and Storvick evaluate certain Yeh-Wiener integrals in terms of Wiener integrals. In particular they obtain the following theorem (this theorem also plays a key role in [3]);

THEOREM A. *Let $\alpha < \gamma \leqslant \beta$ and let f be a real or complex valued functional defined on $C_1[a,b]$ such that $f([(\gamma-\alpha)/2]^{1/2}y)$ is a Wiener measurable functional of y on $C_1[a,b]$. Then $f(x(\cdot,\gamma))$ is a Yeh-Wiener measurable functional of $x(\cdot,\cdot)$ on $C_2[R]$ and*

$$\int_{C_2[R]} f(x(\cdot,\gamma))\,dx = \int_{C_1[a,b]} f([(\gamma-\alpha)/2]^{1/2}y)\,dy \tag{1}$$

where the existence of either integral implies the existence of the other and their equality.

Since one is trying to evaluate Yeh-Wiener integrals it would seem very natural and desirable to assume that the functional $f(x(\cdot,\gamma))$ is a Yeh-Wiener measurable functional of $x(\cdot,\cdot)$ on $C_2[R]$ and conclude that $f([(\gamma-\alpha)/2]^{1/2}y)$ is a Wiener measurable functional of y on $C_1[a,b]$. This would also allow the proof of equation (1) to proceed in the same order as the motivation; where as in [2] the proof had to proceed in the opposite order from the motivation because of the measurability argument. We obtain this result (i.e. the converse of Theorem A) in §2 below. In particular we show that if A is any subset of $C_1[a,b]$ and if $B_A \equiv \{x(\cdot,\cdot) \in C_2[R] \mid x(\cdot,\gamma) \in A\}$ then B_A is a Yeh-Wiener

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measurable subset of $C_2[R]$ if and only if $[2/(\gamma - \alpha)]^{1/2}A$ is a Wiener measurable subset of $C_1[a,b]$.

Let $a = s_0 < s_1 < \cdots < s_m = b$ and $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ be subdivisions of $[a,b]$ and $[\alpha,\beta]$ respectively. Let E be any subset of Euclidean space R^{mn} and let $Q = \{x \in C_2[R] | \langle x(s_1,t_1),\ldots,x(s_m,t_n)\rangle \in E\}$. By the definition of Yeh-Wiener measure, if E is Lebesgue measurable then Q is Yeh-Wiener measurable. In §4 we establish the converse; i.e. if Q is Yeh-Wiener measurable then E is Lebesgue measurable. In an unpublished result Fulton Koehler established this fact for $C_1[a,b]$.

The techniques used in this paper are quite different from the technique used in [2] and [4]. The key to obtaining these "converse theorems" is Lemma 3 in which we show that the outer Yeh-Wiener measure of the set B_A is equal to the outer Wiener measure of the set $[2/(\gamma - \alpha)]^{1/2}A$.

1. **Preliminaries.** In this section, for the convenience of the reader, we will present a brief discussion of Wiener measure on $C_1[a,b]$ and Yeh-Wiener measure on $C_2[R]$.

Let $a = s_0 < s_1 < \cdots < s_m = b$ and let E be a Lebesgue measurable set in Euclidean m-space R^m. Then

$$I = \{y(\cdot) \in C_1[a,b] | \langle y(s_1),\ldots,y(s_m)\rangle \in E\}$$

is called an interval in $C_1[a,b]$. The Wiener measure of the interval I is defined to be

$$m_1(I) \equiv \int_E W(\vec{u};\vec{s})\,d\vec{u}$$

where

$$\begin{aligned} W(\vec{u};\vec{s}) &= W(u_1,\ldots,u_m;s_1,\ldots,s_m) \\ &= \prod_{j=1}^{m} [2\pi(s_j - s_{j-1})]^{-1/2} \exp\left\{\frac{-(u_j - u_{j-1})^2}{2(s_j - s_{j-1})}\right\} \end{aligned}$$

and $u_0 \equiv 0$. This measure is countably additive on the set of all such intervals in $C_1[a,b]$. The outer Wiener measure m_1^* of any subset of $C_1[a,b]$ is now defined in the usual way and the term "Wiener measurable set" will denote any set which is measurable with respect to outer Wiener measure. We let $\mathfrak{M}_1$ denote the class of all Wiener measurable sets and m_1^* restricted to $\mathfrak{M}_1$ will be denoted by m_1. Integration of a functional F with respect to Wiener measure m_1 will be denoted by $\int_{C_1[a,b]} F(y)\,dy$.

Let $a = s_0 < s_1 < \cdots < s_m = b$, $\alpha = t_0 < t_1 < \cdots < t_n = \beta$, and let E be a Lebesgue measurable set in Euclidean space R^{mn}. Then

$$J = \{x(\cdot,\cdot) \in C_2[R] | \langle x(s_1,t_1),\ldots,x(s_m,t_n)\rangle \in E\}$$

is called an interval in $C_2[R]$. The Yeh-Wiener measure of the interval J is defined to be

$$m_2(J) \equiv \int_E W(\vec{u};\vec{s};\vec{t})\,d\vec{u}$$

where

$$
\begin{aligned}
W(\vec{u};\vec{s};\vec{t}) &= W(u_{1,1},\ldots,u_{m,n};s_1,\ldots,s_m;t_1,\ldots,t_n) \\
&= \prod_{j=1}^{m}\prod_{k=1}^{n}[\pi(s_j-s_{j-1})(t_k-t_{k-1})]^{-1/2} \\
&\quad\cdot\exp\left\{\frac{-(u_{j,k}-u_{j-1,k}-u_{j,k-1}+u_{j-1,k-1})^2}{(s_j-s_{j-1})(t_k-t_{k-1})}\right\}
\end{aligned}
\tag{2}
$$

and $u_{0,k} = u_{j,0} = u_{0,0} = 0$ for all j and k. This measure is countably additive on the set of all such intervals in $C_2[R]$. The outer Yeh-Wiener measure m_2^* of any subset of $C_2[R]$ is now defined in the usual way and the term "Yeh-Wiener measurable set" will denote any set which is measurable with respect to outer Yeh-Wiener measure. We let $\mathfrak{M}_2$ denote the class of all Yeh-Wiener measurable sets and m_2^* restricted to $\mathfrak{M}_2$ will be denoted by m_2. Integration of a functional F with respect to Yeh-Wiener measure m_2 will be denoted by $\int_{C_2[R]} F(x)\,dx$.

2. **Some relationships between Yeh-Wiener and Wiener measurability**. Our first theorem in this section (whose rather lengthy proof is given in §3) establishes a relationship between Yeh-Wiener measurability and Wiener measurability of certain related sets.

THEOREM 1. *Let $\alpha < \gamma \leqslant \beta$, let A be any subset of $C_1[a,b]$ and let $B_A \equiv \{x \in C_2[R] \mid x(\cdot,\gamma) \in A\}$. Then $B_A \in \mathfrak{M}_2$ if and only if $[2/(\gamma-\alpha)]^{1/2}A \in \mathfrak{M}_1$. Furthermore the equation*

$$
\begin{aligned}
m_2(B_A) &= \int_{C_2[R]} \chi_A(x(\cdot,\gamma))\,dx \\
&= \int_{C_1[a,b]} \chi_{[2/(\gamma-\alpha)]^{1/2}A}(y)\,dy = m_1([2/(\gamma-\alpha)]^{1/2}A)
\end{aligned}
\tag{3}
$$

holds if either $B_A \in \mathfrak{M}_2$ or $[2/(\gamma-\alpha)]^{1/2}A \in \mathfrak{M}_1$.

THEOREM 2. *Let γ, A and B_A be as in Theorem* 1.
(a) *If $\gamma - \alpha = 2$ then $B_A \in \mathfrak{M}_2 \Leftrightarrow A \in \mathfrak{M}_1$.*
(b) *If $\gamma - \alpha \neq 2$ then $A \in \mathfrak{M}_1 \not\Rightarrow B_A \in \mathfrak{M}_2$.*
(c) *If $\gamma - \alpha \neq 2$ then $B_A \in \mathfrak{M}_2 \not\Rightarrow A \in \mathfrak{M}_1$.*

PROOF. Statement (a) is a special case of Theorem 1. To establish (b) let H be a subset of $C_1[a,b]$ with the property that $m_1(H) = 1$ and $m_1(\lambda H) = 0$ for all real $\lambda \neq \pm 1$. (For the existence of such a set see [1].) Let G be a nonmeasurable subset of $C_1[a,b]$. Then $H \cap G \notin \mathfrak{M}_1$. Let

$$A = [(\gamma-\alpha)/2]^{1/2}(H \cap G).$$

Then A is a null set in $C_1[a,b]$ and so $A \in \mathfrak{M}_1$. But then $[2/(\gamma-\alpha)]^{1/2}A = H \cap G \notin \mathfrak{M}_1$ and so by Theorem 1, $B_A \notin \mathfrak{M}_2$. Hence $A \in \mathfrak{M}_1 \not\Rightarrow B_A \in \mathfrak{M}_2$.

To establish (c) let $A = H \cap G$ where H and G are as above. Then $A \notin \mathfrak{M}_1$. But $[2/(\gamma-\alpha)]^{1/2}A = [2/(\gamma-\alpha)]^{1/2}(H \cap G)$ is a null set in $C_1[a,b]$, hence in $\mathfrak{M}_1$ and so by Theorem 1, $B_A \in \mathfrak{M}_2$. Hence $B_A \in \mathfrak{M}_2 \not\Rightarrow A \in \mathfrak{M}_1$.

THEOREM 3. *Let $\alpha < \gamma \leqslant \beta$. Let f be a real or complex valued functional defined on $C_1[a,b]$ such that $f(x(\cdot,\gamma))$ is a Yeh-Wiener measurable functional of $x(\cdot,\cdot)$ on $C_2[R]$. Then $f([(\gamma-\alpha)/2]^{1/2}y)$ is a Wiener measurable functional of y on $C_1[a,b]$ and*

$$\int_{C_2[R]} f(x(\cdot,\gamma))\,dx = \int_{C_1[a,b]} f([(\gamma-\alpha)/2]^{1/2}y)\,dy \tag{4}$$

where the existence of either integral implies the existence of the other and their equality.

PROOF. *Case* 1. Let $f(y) = \chi_A(y)$ where A is a subset of $C_1[a,b]$ such that $\chi_A(x(\cdot,\gamma))$ is a Yeh-Wiener measurable functional of $x(\cdot,\cdot)$ on $C_2[R]$ (i.e. $f(y) = \chi_A(y)$ where $B_A \in \mathfrak{M}_2$).

Using Theorem 1 we immediately obtain that the functional

$$f([\gamma-\alpha)/2]^{1/2}y) = \chi_A([(\gamma-\alpha)/2]^{1/2}y) = \chi_{[2/(\gamma-\alpha)]^{1/2}A}(y)$$

is Wiener measurable and (4) follows from (3).

Case 2. Let $f(y)$ be a simple functional such that $f(x(\cdot,\gamma))$ is a Yeh-Wiener measurable functional of $x(\cdot,\cdot)$ on $C_2[R]$.

This case follows easily from Case 1 since we can write $f(y)$ in the form $f(y) = \sum_{j=1}^{n} c_j\chi_{A_j}(y)$ where each c_j is real and each $\chi_{A_j}(y)$ is of the type considered in Case 1.

Case 3. Let $f(y)$ be a real nonnegative functional such that $f(x(\cdot,\gamma))$ is a Yeh-Wiener measurable functional of $x(\cdot,\cdot)$ on $C_2[R]$.

In this case $f(y)$ is the limit of a monotone increasing sequence of simple functions and so the desired conclusions follow from Case 2 and the monotone convergence theorem.

Case 4. General case.

This case follows from Case 3 since we can decompose any complex functional into its real and imaginary parts and then into their positive and negative parts.

3. **Proof of Theorem 1.** The proof of Theorem 1 will follow quite readily once we establish three lemmas.

DEFINITION. Let δ be a fixed constant satisfying $0 < \delta < \frac{1}{2}$ and let $h > 0$ be given. Let

$$\begin{aligned} A_h \equiv A_h(\delta) \equiv \{x \in C_2[R] \mid\, & |x(s_2,t_2) - x(s_1,t_1)| \\ & \leqslant h[(s_2-s_1)^2 + (t_2-t_1)^2]^{\delta/2} \\ & \qquad \text{for all } s_1,s_2 \in [a,b] \text{ and } (t_1,t_2) \in [\alpha,\beta]\}. \end{aligned}$$

LEMMA 1. (a) *For any $\varepsilon > 0$ there exists $h_0 > 0$ such that $m_2(A_h^c) < \varepsilon$ for all $h \geqslant h_0$. In fact $m_2(\bigcup_{h=1}^{\infty} A_h) = 1$.* (b) *For each $h > 0$, A_h is compact in the uniform topology.*

PROOF. Statement (a) was established by Yeh [4, Theorem 1]. In [4, Lemma 5] Yeh showed that A_h was compact in itself in the weak topology on $C_2[R]$ i.e. for any sequences $\{x_n\}$ in A_h there exists a subsequence $\{x_{n_k}\}$ which

converges pointwise on R to an element x_0 in A_h. But A_h is equicontinuous and equibounded. Hence by Ascoli's Theorem every sequence of elements from A_h has a subsequence that converges uniformly. But A_h is closed in the uniform topology and hence is sequentially compact in the uniform topology. But the uniform topology is a metric topology and so A_h is compact in the uniform topology on $C_2[R]$.

LEMMA 2. *Let $\alpha < \gamma \leqslant \beta$, let A be any subset of $C_1[a,b]$ and let $B_A = \{x \in C_2[R] | x(\cdot,\gamma) \in A\}$. Let G be any open set in $C_2[R]$ containing B_A. Let $h > 0$ be given. Then there exists an open set U in $C_1[a,b]$ such that $A \subseteq U$ and $(A_h \cap B_U) \subseteq G$ where $B_U = \{x \in C_2[R] | x(\cdot,\gamma) \in U\}$.*

PROOF. *Case* 1. Assume A consists of just one point, say $y_0(\cdot)$. Assume Lemma 2 is false. Then there exists a sequence of points $\{x_n(\cdot,\cdot)\}_{n=1}^{\infty}$ in $A_h - G$ such that

$$\lim_{n\to\infty} x_n(s,\gamma) = y_0(s) \quad \text{for each } s \in [a,b].$$

Since A_h is compact in the uniform topology there exists a subsequence $\{x_{n_k}(\cdot,\cdot)\}_{k=1}^{\infty}$ which converges uniformly, say to $x_0(\cdot,\cdot)$, on R. Thus $\lim_{k\to\infty} x_{n_k}(s,\gamma) = x_0(s,\gamma)$ for all $s \in [a,b]$ and so $x_0(\cdot,\gamma) = y_0(\cdot)$. Hence $x_0 \in B_A$. But G^c is a closed set in $C_2[R]$ and so $x_0 \in G^c \subseteq B_A^c$ which is contrary to $x_0 \in B_A$. Hence Lemma 2 is established when A is a singleton set.

Case 2. General case. Let A be any set in $C_1[a,b]$. Using Case 1 we see that for each point y in A there exists an open set U_y in $C_1[a,b]$ containing y such that $(A_h \cap B_{U_y}) \subseteq G$. Then $U \equiv \bigcup_{y\in A} U_y$ is an open set in $C_1[a,b]$ containing A such that $(A_h \cap B_U) \subseteq G$.

LEMMA 3. *Let γ, A and B_A be as in Lemma 2. Then*

$$m_y^*(B_A) = m_1^*([2/(\gamma-\alpha)]^{1/2}A). \tag{5}$$

PROOF. (i) We will first show that $m_2^*(B_A) \leqslant m_1^*([2/(\gamma-\alpha)]^{1/2}A)$. Let $\hat{A}$ be a subset of $C_1[a,b]$ such that $A \subseteq \hat{A}$, $[2/(\gamma-\alpha)]^{1/2}\hat{A} \in \mathfrak{M}_1$ and

$$m_1([2/(\gamma-\alpha)]^{1/2}\hat{A}) = m_1^*([2/(\gamma-\alpha)]^{1/2}A).$$

Then $B_{\hat{A}} \in \mathfrak{M}_2$, $B_A \subseteq B_{\hat{A}}$ and

$$\begin{aligned} m_2^*(B_A) \leqslant m_2^*(B_{\hat{A}}) &= m_2(B_{\hat{A}}) = m_1([2/(\gamma-\alpha)]^{1/2}\hat{A}) \\ &= m_1^*([2/(\gamma-\alpha)]^{1/2}A). \end{aligned}$$

(ii) Let $\varepsilon > 0$ be given. We need only show that $m_1^*([2/(\gamma-\alpha)]^{1/2}A) \leqslant m_2^*(B_A) + \varepsilon$. First choose $H \in \mathfrak{M}_2$ such that $B_A \subseteq H$ and $m_2^*(B_A) = m_2(H)$. Next choose $h > 0$ so large that $m_2(A_h^c) < \varepsilon/2$. Then

$$m_2(H \cup A_h^c) \leqslant m_2(H) + m_2(A_h^c) < m_2^*(B_A) + \varepsilon/2. \tag{6}$$

Now [4, Lemma 6] let G be an open subset of $C_2[R]$ such that $(H \cup A_h^c) \subseteq G$ and $m_2(G - (H \cup A_h^c)) < \varepsilon/2$. Then

$$m_2(G) < m_2(H \cup A_k^c) + \varepsilon/2. \tag{7}$$

Next, by Lemma 2 above, there exists an open set U in $C_1[a,b]$ such that $A \subseteq U$ and $(A_h \cap B_U) \subseteq G$. But

$$(B_U \cap A_h^c) \subseteq A_h^c \subseteq (H \cup A_h^c) \subseteq G \tag{8}$$

and so $B_U = (B_U \cap A_h^c) \cup (B_U \cap A_h) \subseteq G$. Now since U is an open set in $C_1[a,b]$ it follows that $[2/(\gamma - \alpha)]^{1/2} U \in \mathfrak{M}_1$. But this implies that $B_U \in \mathfrak{M}_2$ and the equality $m_1([2/(\gamma - \alpha)]^{1/2} U) = m_2(B_U)$. Hence using (8), (7), and (6) we obtain

$$\begin{aligned} m_1^*([2/(\gamma - \alpha)]^{1/2} A) &\leqslant m_1([2/(\gamma - \alpha)]^{1/2} U) \\ &= m_2(B_U) \leqslant m_2(G) < m_2^*(B_A) + \varepsilon \end{aligned}$$

which establishes (5).

PROOF (OF THEOREM 1). We only need to show that $B_A \in \mathfrak{M}_2$ implies $[2/(\gamma - \alpha)]^{1/2} A \in \mathfrak{M}_1$. So assume that $B_A \in \mathfrak{M}_2$. Then by Lemma 3

$$m_1^*([2/(\gamma - \alpha)]^{1/2} A) = m_2^*(B_A) = m_2(B_A). \tag{9}$$

Now $B_A \in \mathfrak{M}_2$ implies that $(B_A)^c \in \mathfrak{M}_2$ and since

$$(B_A)^c = B_{A^c} \quad \text{and} \quad ([2/(\gamma - \alpha)]^{1/2} A)^c = [2/(\gamma - \alpha)]^{1/2} A^c,$$

another application of Lemma 3 yields

$$\begin{aligned} m_1^*(([2/(\gamma - \alpha)]^{1/2} A)^c) &= m_1^*([2/(\gamma - \alpha)]^{1/2} A^c) \\ &= m_2^*(B_{A^c}) = m_2^*((B_A)^c) = m_2((B_A)^c). \end{aligned} \tag{10}$$

Hence using (9) and (10) we obtain that

$$m_1^*([2/(\gamma - \alpha)]^{1/2} A) + m_1^*(([2/(\gamma - \alpha)]^{1/2} A)^c) = 1$$

from which it follows that $[2/(\gamma - \alpha)]^{1/2} A \in \mathfrak{M}_1$.

4. Koehler's Theorem for $C_2[R]$.

THEOREM 4. *Let $a = s_0 < s_1 < \cdots < s_m = b$ and $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ be subdivisions of $[a,b]$ and $[\alpha,\beta]$ respectively. Let E be any subset of Euclidean space R^{mn} and let*

$$Q = \{x \in C_2[R] \mid \langle x(s_1,t_1), \ldots, x(s_m,t_n)\rangle \in E\}. \tag{11}$$

If $Q \in \mathfrak{M}_2$ then E is a Lebesgue measurable subset of R^{mn}.

COROLLARY. *Let f: $R^{mn} \to C$ and let F: $C_2[R] \to C$ be defined by $F(x) = f(x(s_1,t_1), \ldots, x(s_m,t_n))$. If $F(x)$ is a Yeh-Wiener measurable functional on $C_2[R]$ then $f(u_{1,1}, \ldots, u_{m,n})$ is Lebesgue measurable on R^{mn}.*

The proof of Theorem 4 follows quite easily once Lemma 5 below is established. We will omit the proofs of Lemmas 4 and 5 below since they are similar to the proofs given above for Lemmas 2 and 3 respectively.

LEMMA 4. *Let E and Q be as in Theorem 4. Let G be any open set in $C_2[R]$ containing Q. Let $h > 0$ be given. Then there exists an open set U in R^{mn} containing E such that*

$$\{x \in C_2[R] \mid x \in A_h \text{ and } \langle x(s_1, t_1), \ldots, x(s_m, t_n)\rangle \in U\} \subseteq G. \tag{12}$$

DEFINITION. We define a probability measure v on the Lebesgue measurable subsets E of R^{mn} by

$$v(E) \equiv \int_E W(\vec{u}\,;\vec{s}\,;\vec{t}\,)\,d\vec{u}$$

where $W(\vec{u}\,;\vec{s}\,;\vec{t}\,)$ is given by (2). Let v^* be the usual regular outer measure based on v such that v^* is defined for all subsets of R^{mn}.

LEMMA 5. *Let E be any subset of R^{mn} and let Q be defined by* (11). *Then* $m_2^*(Q) = v^*(E)$.

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