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## SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

# ON A LIPSCHITZ ESTIMATE FOR CONFORMAL MAPS IN THE PLANE

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By combining a weak form of Koebe's $\frac{1}{4}$-Theorem with elementary facts about subharmonic functions, one obtains a brief proof of the following statement.

THEOREM. *Suppose $D$ and $D^*$ are bounded domains in $\mathbf{C}$ with $C^2$ boundary. Then any conformal map $f$: $D \to D^*$ is Lipschitz, i.e. there is $K < \infty$, such that $|f(z) - f(\zeta)| \leqslant K|z - \zeta|$ for $z, \zeta \in D$.*

By a $C^2$ boundary I mean that $\partial D$ is a not necessarily connected, compact, 1-dimensional $C^2$ submanifold of $\mathbf{C}$, or, equivalently, that $\partial D$ consists of finitely many disjoint closed Jordan curves whose parametrizations in terms of arclength are twice continuously differentiable.

Stronger results, with more complicated proofs, have been known for a long time; for a more recent proof, one may consult Warschawski [4], where one can also find references to the older literature. I thank Professor H. Grunsky for pointing out this reference to me.

PROOF. Fix $z \in D$ and restrict $f$ to the disc with center $z$ and radius $d(z, \partial D)$. By Koebe's Theorem [1, Chapter IV, Satz 59], the image of this disc contains a disc with center $f(z)$ and radius $\rho \cdot |f'(z)| \cdot d(z, \partial D)$, where $\rho > 0$ is independent of $z$. This implies

$$(*) \qquad |f'(z)| \leqslant 1/\rho \, d(f(z), \partial D^*) \cdot d(z, \partial D)^{-1} \quad \text{for } z \in D.$$

Choose a $C^2$ function $\phi$ in a neighborhood of $\overline{D}$, such that $\phi < 0$ on $D$, $\phi = 0$ on $\partial D$ and grad $\phi \neq 0$ on $\partial D$. Locally, the existence of $\phi$ is an immediate consequence of the definition of a $C^2$ submanifold, and the global existence follows by a partition of unity argument. Replacing $\phi$ by $\exp(A\phi) - 1$ with sufficiently large $A > 0$, one can also assume that $\partial^2\phi/\partial z \partial \bar{z} > 0$ near $\partial D$. Moreover, there is $k < \infty$ such that

$$(**) \qquad |\phi(z)| \leqslant k \cdot d(z, \partial D) \quad \text{for } z \in D.$$

As suggested by Henkin [2], one now uses the following lemma.

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LEMMA (CF. [3]). *Suppose $D \subset \mathbf{C}$ has $C^2$ boundary. Let $u$ be continuous on $\overline{D}$, $u < 0$ on $D$, $u = 0$ on $\partial D$, and $u$ is subharmonic on $U \cap D$ for some neighborhood $U$ of $\partial D$. Then there is $c > 0$, such that*

$$|u(w)| \geqslant c \cdot d(w, \partial D) \quad \textit{for } w \in D.$$

Applying this Lemma to $\phi \circ f^{-1}$ and $D^*$, together with (**), one obtains $d(f(z), \partial D^*) \leqslant k/c \cdot d(z, \partial D)$. This estimate and (*) show that $|f'|$ is bounded, and hence $f$ is Lipschitz. Q.E.D.

PROOF OF THE LEMMA. Choose $r > 0$, such that for each $\zeta \in \partial D$ the interior $\Delta_\zeta(r)$ of the circle of radius $r$ through $\zeta$ with center on the inner normal to $\partial D$ at $\zeta$ lies in $U \cap D$. Let $M = \sup\{u(z)\colon z \in D, d(z, \partial D) \geqslant r\}$; $M$ is $< 0$. For $z \in D$ with $d(z, \partial D) < r$ choose $\zeta \in \partial D$ with $|z - \zeta| = d(z, \partial D)$. Let $P_r$ be the Poisson kernel for the disc $\Delta_\zeta(r)$, and let $d\sigma$ be arclength on $\partial\Delta_\zeta(r)$. Then

$$u(z) \leqslant \int_{\partial\Delta_\zeta} u(\lambda) P_r(z, \lambda)\, d\sigma(\lambda) \leqslant M \int_{\partial\Delta_\zeta \cap \{z : d(z, \partial D) \geqslant r\}} P_r(z, \lambda)\, d\sigma(\lambda),$$

and the Lemma now follows directly from properties of $P_r$.

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