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GENERALIZED CENTER AND HYPERCENTER OF A FINITE GROUP

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ABSTRACT. The generalized center of a group G is defined to be the subgroup generated by all elements g of G such that $\langle g\rangle P = P\langle g\rangle$ for all Sylow subgroups P of G. This generalizes the concept of the center and quasicenter and leads to the notion of the generalized hypercenter which is defined in the same way as the hypercenter and hyperquasicenter. It is shown that the generalized center is nilpotent and the generalized hypercenter is supersolvable (in fact, the generalized hypercenter is contained in the intersection of the maximal supersolvable subgroups).

Generalizing the notion of the center, Ore [10] defined the quasicenter $Q(G)$ of a group G to be the subgroup generated by all elements g of G such that $\langle g\rangle H = H\langle g\rangle$ for all $H \leqslant G$. Mukherjee [9] determined its structure. He also defined and studied the hyperquasicenter $Q^*(G)$ of G which generalizes the concept of the hypercenter. Here we define and investigate another center and hypercenter of a finite group.

We say, following Kegel [7], that a subgroup of G is π-quasinormal in G if it permutes with every Sylow subgroup of G and define the generalized center $Z_{\mathrm{Gn}}(G)$ of G to be the subgroup generated by all elements g of G such that $\langle g\rangle$ is π-quasinormal in G. This leads to the definition of the generalized hypercenter $Z^*_{\mathrm{Gn}}(G)$. Let $(Z_{\mathrm{Gn}}(G))_0 = 1$ and $(Z_{\mathrm{Gn}}(G))_{i+1}/(Z_{\mathrm{Gn}}(G))_i$ be the generalized center of $G/(Z_{\mathrm{Gn}}(G))_i$. This yields an ascending chain of characteristic subgroups

$$1 = (Z_{\mathrm{Gn}}(G))_0 < (Z_{\mathrm{Gn}}(G))_1 < (Z_{\mathrm{Gn}}(G))_2 < \cdots < (Z_{\mathrm{Gn}}(G))_m = Z^*_{\mathrm{Gn}}(G).$$

The terminal member $Z^*_{\mathrm{Gn}}(G)$ is called the generalized hypercenter.

The main results are: (1) If $\langle g\rangle$ is π-quasinormal in G, so is every subgroup of $\langle g\rangle$; (2) $Z_{\mathrm{Gn}}(G)$ is nilpotent; (3) If $G = H \times K$, then

$$Z_{\mathrm{Gn}}(G) = Z_{\mathrm{Gn}}(H) \times Z_{\mathrm{Gn}}(K) \quad \text{and} \quad Z^*_{\mathrm{Gn}}(G) = Z^*_{\mathrm{Gn}}(H) \times Z^*_{\mathrm{Gn}}(K);$$

(4) $Z^*_{\mathrm{Gn}}(G)$ is supersolvable; (5) $Z^*_{\mathrm{Gn}}(G)$ is contained in the intersection of the maximal supersolvable subgroups of G; and (6) $Z^*_{\mathrm{Gn}}(G)$ is the product of all generalized hypercentral subgroups of G.

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Throughout, only *finite* groups are considered.

1. **Generalized center.**

DEFINITIONS. Subgroups H and K of G permute if $HK = KH$. A subgroup of G is called π-quasinormal (quasinormal) in G if it permutes with all Sylow subgroups (all subgroups) of G. The quasicenter $Q(G)$ of G is the subgroup generated by all elements g of G such that $\langle g \rangle$ is quasinormal in G.

The following results on π-quasinormality are frequently used later.

(1.1) Kegel [7]. If $H \leqslant K \leqslant G$ and H is π-quasinormal in G, then H is π-quasinormal in K.

(1.2) Kegel [7]. A π-quasinormal subgroup of G is subnormal in G.

(1.3) Kegel [7]. If θ is a homomorphism from G onto G^θ and H is a π-quasinormal subgroup of G, then H^θ is a π-quasinormal subgroup of G^θ.

(1.4) Let p be a prime dividing $|G|$ and H be a p-subgroup of G. If H is π-quasinormal in G, then $H \lhd HG_q$ for every Sylow q-subgroup G_q of G with $q \neq p$.

PROOF. Since $HG_q = G_qH$, HG_q is a subgroup. By (1.1), H is π-quasinormal in HG_q. Hence by (1.2), H is a subnormal Sylow p-subgroup of HG_q. This implies that H is normal in HG_q.

DEFINITION. The generalized center $Z_{\mathrm{Gn}}(G)$ of a group G is the subgroup generated by all elements g of G such that $\langle g \rangle$ is π-quasinormal in G. Such elements shall be called generalized central elements of G.

REMARK. $Z_{\mathrm{Gn}}(G)$ is a characteristic subgroup of G which contains the quasicenter $Q(G)$ and the center $Z(G)$ of G. The following example shows that $Z_{\mathrm{Gn}}(G)$ can be nontrivial even though $Q(G)$ is trivial.

EXAMPLE 1. Let G be the symmetric group S_3 wreathed by the cyclic group C_3 of order 3. That is, G is the direct product of three copies of S_3 extended by an automorphism of order 3 which permutes the copies in a cycle. For this group, $Q(G) = 1$ but $Z_{\mathrm{Gn}}(G)$ has order 27.

Subgroups of a normal cyclic subgroup are normal in the group. A similar result holds for the subgroups of a π-quasinormal cyclic subgroup.

THEOREM 1.5. *If $\langle g \rangle$ is π-quasinormal in G, then every subgroup of $\langle g \rangle$ is also π-quasinormal in G.*

PROOF. Let $\langle g^n \rangle$ be a subgroup of $\langle g \rangle$, where n is an integer. To prove the theorem, we must show that $\langle g^n \rangle G_p = G_p \langle g^n \rangle$ for an arbitrary but fixed Sylow p-subgroup G_p of G. By (1.2), $\langle g \rangle$ is subnormal in G. Hence $\langle g^n \rangle$ is subnormal in G. Let P be the Sylow p-subgroup of $\langle g^n \rangle$ and K be the p-complement of $\langle g^n \rangle$. Now, since P is a subnormal p-subgroup of G, it follows that $P \leqslant G_p$ and therefore $PG_p = G_pP = G_p$. Let H be the p-complement of $\langle g \rangle$. Then H is subnormal in G and hence H is subnormal in $\langle g \rangle G_p$. This means that H is a subnormal Hall subgroup of $\langle g \rangle G_p$, which implies that $H \lhd \langle g \rangle G_p$. But $K \leqslant H$ and H is cyclic. Hence $K \lhd \langle g \rangle G_p$ and so $KG_p = G_pK$. Since $\langle g^n \rangle = P \times K$, it follows that $\langle g^n \rangle$ and G_p permute.

REMARK. Using (1.3), one can verify that if $G \cong \overline{G}$ under the isomorphism θ, then $(Z_{\mathrm{Gn}}(G))^\theta = Z_{\mathrm{Gn}}(\overline{G})$.

The following example shows that every subgroup of $Z_{\mathrm{Gn}}(G)$ need not be

π-quasinormal in G and that if $\langle a\rangle$ and $\langle b\rangle$ are π-quasinormal in G, then $\langle ab\rangle$ is not always π-quasinormal in G.

EXAMPLE 2. Let $G = \langle a, b\rangle\langle x\rangle$ with $a^3 = b^3 = 1$, $ab = ba$, $a^x = a^2$, $b^x = b$ and $x^2 = 1$. Clearly, $Z_{\mathrm{Gn}}(G) = \langle a, b\rangle$. But $\langle ab\rangle$ is not π-quasinormal in G since $\langle ab\rangle\langle x\rangle \neq \langle x\rangle\langle ab\rangle$.

PROPOSITION 1.6. *If $G = H \times K$, then $Z_{\mathrm{Gn}}(G) = Z_{\mathrm{Gn}}(H) \times Z_{\mathrm{Gn}}(K)$.*

PROOF. First we show that $Z_{\mathrm{Gn}}(H) \times Z_{\mathrm{Gn}}(K) \leqslant Z_{\mathrm{Gn}}(G)$. Let h be an element of H such that $\langle h\rangle$ is π-quasinormal in H and G_p be a Sylow p-subgroup of G. Since $G_p = H_p \times K_p$ for some H_p and K_p and $\langle h\rangle$ is centralized by K, it follows that $\langle h\rangle G_p = G_p\langle h\rangle$. Hence, $\langle h\rangle$ is π-quasinormal in G and so $h \in Z_{\mathrm{Gn}}(G)$. Thus $Z_{\mathrm{Gn}}(H) \leqslant Z_{\mathrm{Gn}}(G)$. Similarly, $Z_{\mathrm{Gn}}(K) \leqslant Z_{\mathrm{Gn}}(G)$.

Next to show that $Z_{\mathrm{Gn}}(G) \leqslant Z_{\mathrm{Gn}}(H) \times Z_{\mathrm{Gn}}(K)$, let g be an element of G such that $\langle g\rangle$ is π-quasinormal in G. Let P be the Sylow p-subgroup of $\langle g\rangle$. Clearly, $P = \langle hk\rangle$ for some p-elements $h \in H$ and $k \in K$. We will show that $\langle h\rangle$ is π-quasinormal in H. For this, let H_q be any Sylow q-subgroup of H, $q \neq p$ and x be an element of H_q. By Theorem 1.5, P is π-quasinormal in G. Hence it follows from (1.4) that $P \lhd PG_q$ for every Sylow q-subgroup G_q of G, $q \neq p$. In particular, $x^{-1}(hk)x = (hk)^n = h^n k^n$ for some integer n. This implies that $x^{-1}hx = h^n$ and so $\langle h\rangle$ permutes with every H_q for primes $q \neq p$. Since P is π-quasinormal in G, $P = \langle hk\rangle \leqslant \cap G_p = \cap(H_p \times K_p)$. Hence h belongs to every Sylow p-subgroup H_p of H. Consequently, $\langle h\rangle H_p = H_p\langle h\rangle = H_p$ which implies that $\langle h\rangle$ is π-quasinormal in H and so $h \in Z_{\mathrm{Gn}}(H)$. Similarly $k \in Z_{\mathrm{Gn}}(K)$. Thus $P \leqslant Z_{\mathrm{Gn}}(H) \times Z_{\mathrm{Gn}}(K)$. Since $\langle g\rangle$ is the direct product of its Sylow subgroups, it follows that $\langle g\rangle \leqslant Z_{\mathrm{Gn}}(H) \times Z_{\mathrm{Gn}}(K)$. This completes the proof.

REMARK. Note that if G is nilpotent, then $Z_{\mathrm{Gn}}(G) = G$ but the quasicenter $Q(G)$ is not necessarily G.

To determine the structure of $Z_{\mathrm{Gn}}(G)$, we need the following:

LEMMA 1.7. *Every Sylow subgroup of $Z_{\mathrm{Gn}}(G)$ of a group G is generated by generalized central elements of G.*

PROOF. Let $|Z_{\mathrm{Gn}}(G)| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, where $p_1, p_2, \ldots, p_n$ are primes. For $1 \leqslant i \leqslant n$, denote by S_{p_i} the subgroup generated by all p_i-elements of G that are generalized central elements of G. Clearly, each S_{p_i} is π-quasinormal in G and hence is subnormal in G. Furthermore, since a generalized central p_i-element of G belongs to every Sylow p_i-subgroup G_{p_i} of G, it follows that $S_{p_i} \leqslant \cap G_{p_i}$. Therefore, S_{p_i} is a p_i-subgroup of G.

Let p_j and p_k be any two different primes dividing the order of $Z_{\mathrm{Gn}}(G)$. Then, by the preceding paragraph and (1.4), $S_{p_j} \lhd S_{p_j} G_{p_k}$. Hence S_{p_j} and S_{p_k} permute. Thus, $S_{p_j} S_{p_k}$ is a subgroup for all primes p_j and p_k with $p_j \neq p_k$ and $1 \leqslant j, k \leqslant n$. Therefore, $S_{p_1} S_{p_2} \cdots S_{p_n}$ is a subgroup which is contained in $Z_{\mathrm{Gn}}(G)$. Now, let g be any generalized central element of G. Since $\langle g\rangle$ is the direct product of its Sylow subgroups each of which is cyclic and π-quasinormal in G, it follows that every Sylow subgroup of $\langle g\rangle$ is contained in some S_{p_i}. Hence $\langle g\rangle \leqslant S_{p_1} S_{p_2} \cdots S_{p_n}$, which implies that $Z_{\mathrm{Gn}}(G)$

$\leqslant S_{p_1} S_{p_2} \cdots S_{p_n}$. Thus $Z_{\mathrm{Gn}}(G) = S_{p_1} S_{p_2} \cdots S_{p_n}$. From this, the assertion in the lemma follows immediately.

THEOREM 1.8. *$Z_{\mathrm{Gn}}(G)$ of a group G is nilpotent.*

PROOF. Let P be a Sylow subgroup of $Z_{\mathrm{Gn}}(G)$. By Lemma 1.7, P is subnormal (in fact π-quasinormal) in G. Hence P is subnormal in $Z_{\mathrm{Gn}}(G)$ and so $P \lhd Z_{\mathrm{Gn}}(G)$. Thus $Z_{\mathrm{Gn}}(G)$ is nilpotent.

REMARK. Clearly, every Sylow subgroup of $Z_{\mathrm{Gn}}(G)$ is characteristic in G and G is nilpotent if and only if $G = Z_{\mathrm{Gn}}(G)$.

2. **Generalized hypercenter.**

DEFINITION. For a group G, let $(Z_{\mathrm{Gn}}(G))_0 = 1$ and $(Z_{\mathrm{Gn}}(G))_{i+1}/(Z_{\mathrm{Gn}}(G))_i$ be the generalized center of $G/(Z_{\mathrm{Gn}}(G))_i$. The generalized hypercenter $Z^*_{\mathrm{Gn}}(G)$ of G is the terminal member of the chain of characteristic subgroups

$$1 = (Z_{\mathrm{Gn}}(G))_0 < Z_{\mathrm{Gn}}(G) = (Z_{\mathrm{Gn}}(G))_1 < (Z_{\mathrm{Gn}}(G))_2 < \cdots < (Z_{\mathrm{Gn}}(G))_m = Z^*_{\mathrm{Gn}}(G).$$

REMARK. $Z^*_{\mathrm{Gn}}(G)$ contains the hypercenter and hyperquasicenter of G. Example 1 of §1 shows that the generalized hypercenter of a group can be nontrivial even though its hyperquasicenter is trivial. Also note that $Z^*_{\mathrm{Gn}}(G)$ is not necessarily nilpotent as shown by the symmetric group on 3 letters. However, we shall show that $Z^*_{\mathrm{Gn}}(G)$ is supersolvable.

THEOREM 2.1. *For a group G, $Z^*_{\mathrm{Gn}}(G) = \cap\{N \mid N \lhd G$ and $Z_{\mathrm{Gn}}(G/N) = \bar{1}\}$.*

PROOF. Consider the chain

$$1 = (Z_{\mathrm{Gn}}(G))_0 < (Z_{\mathrm{Gn}}(G))_1 < \cdots < (Z_{\mathrm{Gn}}(G))_m = Z^*_{\mathrm{Gn}}(G)$$

and let $T = \cap\{N \mid N \lhd G$ and $Z_{\mathrm{Gn}}(G/N) = \bar{1}\}$. Then $T \leqslant Z^*_{\mathrm{Gn}}(G)$ since $Z_{\mathrm{Gn}}(G/Z^*_{\mathrm{Gn}}(G)) = \bar{1}$. To prove $Z^*_{\mathrm{Gn}}(G) \leqslant T$, let K be any element of $\{N \mid N \lhd G$ and $Z_{\mathrm{Gn}}(G/N) = \bar{1}\}$. It is easy to see that $(Z_{\mathrm{Gn}}(G))_1 = Z_{\mathrm{Gn}}(G) \leqslant K$. Next we assume that $(Z_{\mathrm{Gn}}(G))_i \leqslant K$, $1 \leqslant i < m$, and show that $(Z_{\mathrm{Gn}}(G))_{i+1} \leqslant K$. Let $g(Z_{\mathrm{Gn}}(G))_i$ be any generalized central element of $G/(Z_{\mathrm{Gn}}(G))_i$. Since G/K is a homomorphic image of $G/(Z_{\mathrm{Gn}}(G))_i$, it follows that gK is a generalized central element of G/K. But $Z_{\mathrm{Gn}}(G/K) = \bar{1}$. Hence $g \in K$ which implies that

$$Z_{\mathrm{Gn}}(G/(Z_{\mathrm{Gn}}(G))_i) = (Z_{\mathrm{Gn}}(G))_{i+1}/(Z_{\mathrm{Gn}}(G))_i \leqslant K/(Z_{\mathrm{Gn}}(G))_i.$$

Thus $(Z_{\mathrm{Gn}}(G))_{i+1} \leqslant K$ and so $Z^*_{\mathrm{Gn}}(G) \leqslant K$. The rest is obvious.

The next proposition follows from the preceding theorem.

PROPOSITION 2.2. *Let N be a normal subgroup of G. Then $Z^*_{\mathrm{Gn}}(G)N/N \leqslant Z^*_{\mathrm{Gn}}(G/N)$.*

The inclusion here can be proper as shown by the alternating group of degree 4 and its Sylow 2-subgroup.

PROPOSITION 2.3. *Let T be a normal subgroup of G such that $T \leqslant Z^*_{\mathrm{Gn}}(G)$. Then $Z^*_{\mathrm{Gn}}(G/T) = Z^*_{\mathrm{Gn}}(G)/T$.*

PROOF. Consider the chain

$$\bar{1} = T/T < W_1/T < W_2/T < \cdots < W_m/T = Z^*_{\mathrm{Gn}}(G/T),$$

where

$$W_1/T = Z_{\mathrm{Gn}}(G/T) \quad \text{and} \quad W_i/T/W_{i-1}/T = Z_{\mathrm{Gn}}(G/T/W_{i-1}/T)$$

for $i = 2, 3, \ldots, m$. Since $T \leqslant Z^*_{\mathrm{Gn}}(G)$, Proposition 2.2 implies that $Z^*_{\mathrm{Gn}}(G) \leqslant W_m$. To prove $W_m \leqslant Z^*_{\mathrm{Gn}}(G)$, we first show that $W_1 \leqslant Z^*_{\mathrm{Gn}}(G)$. For this, let gT be any generalized central element of G/T. Then $gZ^*_{\mathrm{Gn}}(G)$ is a generalized central element of $G/Z^*_{\mathrm{Gn}}(G)$ since $T \leqslant Z^*_{\mathrm{Gn}}(G)$. Hence $g \in Z^*_{\mathrm{Gn}}(G)$ which implies that $W_1 \leqslant Z^*_{\mathrm{Gn}}(G)$. We now assume that $W_i \leqslant Z^*_{\mathrm{Gn}}(G)$, $2 \leqslant i < m$, and will show that $W_{i+1} \leqslant Z^*_{\mathrm{Gn}}(G)$.

Since $G/T/W_i/T \cong G/W_i$, it follows easily that $Z_{\mathrm{Gn}}(G/W_i) = W_{i+1}/W_i$. Now replacing T by W_i in the argument used above, one can show that $W_{i+1} \leqslant Z^*_{\mathrm{Gn}}(G)$. Hence $W_m \leqslant Z^*_{\mathrm{Gn}}(G)$.

Let H be a subgroup of G. In general, there is no relationship between $Z^*_{\mathrm{Gn}}(G)$ and $Z^*_{\mathrm{Gn}}(H)$. But if $Z^*_{\mathrm{Gn}}(G) \leqslant H$, then we have

PROPOSITION 2.4. *If H is a subgroup of G and $Z^*_{\mathrm{Gn}}(G) \leqslant H$, then $Z^*_{\mathrm{Gn}}(G) \leqslant Z^*_{\mathrm{Gn}}(H)$. In particular, $Z^*_{\mathrm{Gn}}(Z^*_{\mathrm{Gn}}(G)) = Z^*_{\mathrm{Gn}}(G)$.*

One can show that if G and $\bar{G}$ are isomorphic under the map α, then $Z^*_{\mathrm{Gn}}(\bar{G})$ is the image of $Z^*_{\mathrm{Gn}}(G)$ under α. Using this and Proposition 1.6, we prove the following.

PROPOSITION 2.5. *If $G = H \times K$, then $Z^*_{\mathrm{Gn}}(G) = Z^*_{\mathrm{Gn}}(H) \times Z^*_{\mathrm{Gn}}(K)$.*

PROOF. We use induction on $|G|$. Clearly, we may assume that $Z_{\mathrm{Gn}}(G) = Z_{\mathrm{Gn}}(H) \times Z_{\mathrm{Gn}}(K) \neq 1$. By induction and Proposition 2.3,

$$\begin{aligned} Z^*_{\mathrm{Gn}}(H/Z_{\mathrm{Gn}}(H) \times K/Z_{\mathrm{Gn}}(K)) &= Z^*_{\mathrm{Gn}}(H/Z_{\mathrm{Gn}}(H)) \times Z^*_{\mathrm{Gn}}(K/Z_{\mathrm{Gn}}(K)) \\ &= Z^*_{\mathrm{Gn}}(H)/Z_{\mathrm{Gn}}(H) \times Z^*_{\mathrm{Gn}}(K)/Z_{\mathrm{Gn}}(K). \end{aligned}$$

Let $G/Z_{\mathrm{Gn}}(G)$ and $H/Z_{\mathrm{Gn}}(H) \times K/Z_{\mathrm{Gn}}(K)$ be isomorphic under the map θ. Then,

$$\begin{aligned} &Z^*_{\mathrm{Gn}}(H)/Z_{\mathrm{Gn}}(H) \times Z^*_{\mathrm{Gn}}(K)/Z_{\mathrm{Gn}}(K) \\ &\qquad = (Z^*_{\mathrm{Gn}}(G/Z_{\mathrm{Gn}}(G)))^{\theta} = (Z^*_{\mathrm{Gn}}(G)/Z_{\mathrm{Gn}}(G))^{\theta}. \end{aligned}$$

Now apply θ^{-1} to get the desired result.

The next result is needed to determine the structure of $Z^*_{\mathrm{Gn}}(G)$.

THEOREM 2.6. *$Z^*_{\mathrm{Gn}}(G)$ of a group G has the Sylow tower property of supersolvable groups.*

PROOF. We use induction on $|G|$. Thus, in view of Proposition 2.4, we may

assume that $Z^*_{\mathrm{Gn}}(G) = G$. Let p be the largest prime divisor of $|G|$ and G_p be a Sylow p-subgroup of G. We shall show that $G_p \lhd G$.

If p does not divide $|G/Z_{\mathrm{Gn}}(G)|$, then $G_p \leqslant Z_{\mathrm{Gn}}(G)$. But $Z_{\mathrm{Gn}}(G)$ is nilpotent and so $G_p \lhd G$. On the other hand, if p divides $|G/Z_{\mathrm{Gn}}(G)|$, then by induction $(G/Z_{\mathrm{Gn}}(G))_p$ is normal in $G/Z_{\mathrm{Gn}}(G)$ since $Z^*_{\mathrm{Gn}}(G/Z_{\mathrm{Gn}}(G)) = G/Z_{\mathrm{Gn}}(G)$. Hence $G_p Z_{\mathrm{Gn}}(G) \lhd G$. But the Sylow subgroups of $Z_{\mathrm{Gn}}(G)$ are normal in G and so $G_p Z_{\mathrm{Gn}}(G) = G_p L \lhd G$, where L is the p-complement of $Z_{\mathrm{Gn}}(G)$. To prove that $G_p \lhd G$, it suffices to show that $L \leqslant N_G(G_p)$ since $G = G_p L N_G(G_p) = L N_G(G_p)$ by Frattini's Lemma. For this, let x be any q-element of G for a prime $q \neq p$ such that $\langle x\rangle$ is π-quasinormal in G. By (1.4), $\langle x\rangle \lhd \langle x\rangle G_p$. Let g be any element of G_p. Then $\langle x\rangle\langle g\rangle$ is a supersolvable subgroup of G and $\langle x\rangle \lhd \langle x\rangle\langle g\rangle$. Since $p > q$, $\langle g\rangle$ is also normal in $\langle x\rangle\langle g\rangle$. Hence g and x centralize each other and so $x \in N_G(G_p)$. It follows from Lemma 1.7 that $L \leqslant N_G(G_p)$. Thus $G_p \lhd G$.

Now consider G/G_p. By induction, $Z^*_{\mathrm{Gn}}(G/G_p) = G/G_p$ has the Sylow tower property of supersolvable groups, which means that $G = Z^*_{\mathrm{Gn}}(G)$ has this property. This proves the theorem.

We are now ready to prove

THEOREM 2.7. *$Z^*_{\mathrm{Gn}}(G)$ of a group G is supersolvable.*

PROOF. We use induction on $|G|$. Since $Z^*_{\mathrm{Gn}}(Z^*_{\mathrm{Gn}}(G)) = Z^*_{\mathrm{Gn}}(G)$, we may assume that $Z^*_{\mathrm{Gn}}(G) = G$. This means that every factor group G/K for $K \neq 1$ is supersolvable by induction because $Z^*_{\mathrm{Gn}}(G/K) = Z^*_{\mathrm{Gn}}(G)/K = G/K$.

If the Frattini subgroup $\Phi(G)$ of G is not identity, then $G/\Phi(G)$ is supersolvable and, therefore, G is supersolvable. So we may assume that $\Phi(G) = 1$. Let p be the largest prime divisor of $|G|$. By Theorem 2.6, the Sylow p-subgroup P of G is normal in G. Hence $\Phi(P) \leqslant \Phi(G)$ and so $\Phi(P) = 1$. This means that P is elementary abelian. Let M be a maximal subgroup of G. We will show that $[G\colon M]$ is a prime.

By Theorem 2.6, G is solvable. Hence $[G\colon M]$ is a power of a prime. If $[G\colon M]$ is not a power of p, then $P \leqslant M$ and, since G/P is supersolvable, $[G/P\colon M/P] = [G\colon M]$ is a prime. On the other hand, if $[G\colon M]$ is a power of p, we consider $Z_{\mathrm{Gn}}(G)$ and proceed as follows: Clearly, $Z_{\mathrm{Gn}}(G) \neq 1$. If a prime $q \neq p$ divides $|Z_{\mathrm{Gn}}(G)|$, then the Sylow q-subgroup $\overline{Q}$ of $Z_{\mathrm{Gn}}(G)$ is normal in G and so $\overline{Q} \leqslant M$, which in turn yields that $[G\colon M]$ is a prime. Hence we may assume that $Z_{\mathrm{Gn}}(G)$ is a p-subgroup. Since $Z_{\mathrm{Gn}}(G)$ is generated by generalized central elements of G and the powers of a generalized central element of G are generalized central elements of G, it follows that G contains generalized central elements of order p. Let N be the subgroup generated by all such elements. Then $N \lhd G$ because a conjugate of a π-quasinormal subgroup is a π-quasinormal subgroup. If $N \leqslant M$, then, as before, $[G\colon M]$ is a prime. On the other hand, if $N \nleqslant M$, then there is an element y of order p such that $\langle y\rangle$ is π-quasinormal in G and $y \notin M$. Since P is abelian, $\langle y\rangle$ is a normal subgroup of P. By (1.4), p'-elements of G normalize $\langle y\rangle$. Hence $\langle y\rangle \lhd G$ and so $M\langle y\rangle$ is a subgroup. This implies that $M\langle y\rangle = G$. Therefore, $[G\colon M] = |\langle y\rangle| = p$, a prime. Now $G = Z^*_{\mathrm{Gn}}(G)$ is super-

solvable by a theorem of Huppert. This completes the proof.

One can easily verify the next result.

THEOREM 2.8. *Let G be a group. Then*:

(i) *G is supersolvable if and only if $G = Z^*_{\mathrm{Gn}}(G)$.*

(ii) *G is supersolvable if and only if $G/Z^*_{\mathrm{Gn}}(G)$ is supersolvable.*

THEOREM 2.9. *If S is a supersolvable subgroup of a group G, then $SZ^*_{\mathrm{Gn}}(G)$ is supersolvable.*

PROOF. We use induction on $|G|$. If $SZ^*_{\mathrm{Gn}}(G) \leqslant G$, then $SZ^*_{\mathrm{Gn}}(SZ^*_{\mathrm{Gn}}(G))$ is supersolvable by induction. Proposition 2.4 yields that $Z^*_{\mathrm{Gn}}(G) \leqslant Z^*_{\mathrm{Gn}}(SZ^*_{\mathrm{Gn}}(G))$ and so $SZ^*_{\mathrm{Gn}}(G)$ is supersolvable. On the other hand, if $SZ^*_{\mathrm{Gn}}(G) = G$, then, since

$$G/Z^*_{\mathrm{Gn}}(G) = SZ^*_{\mathrm{Gn}}(G)/Z^*_{\mathrm{Gn}}(G) \cong S/S \cap Z^*_{\mathrm{Gn}}(G)$$

and $Z^*_{\mathrm{Gn}}(G/Z^*_{\mathrm{Gn}}(G)) = 1$, it follows that $Z^*_{\mathrm{Gn}}(S/S \cap Z^*_{\mathrm{Gn}}(G)) = 1$. But $S/S \cap Z^*_{\mathrm{Gn}}(G)$ is supersolvable and so

$$Z^*_{\mathrm{Gn}}(S/S \cap Z^*_{\mathrm{Gn}}(G)) = S/S \cap Z^*_{\mathrm{Gn}}(G).$$

Hence $S/S \cap Z^*_{\mathrm{Gn}}(G) = 1$. This implies that $S \leqslant Z^*_{\mathrm{Gn}}(G)$, which means that $SZ^*_{\mathrm{Gn}}(G) = Z^*_{\mathrm{Gn}}(G)$. Thus $SZ^*_{\mathrm{Gn}}(G)$ is supersolvable.

As a consequence of the last theorem, we have

THEOREM 2.10. *$Z^*_{\mathrm{Gn}}(G)$ of a group G is contained in the intersection of the maximal supersolvable subgroups of G.*

PROOF. Let M be any maximal supersolvable subgroup of G. By Theorem 2.9, $MZ^*_{\mathrm{Gn}}(G)$ is supersolvable. Hence either $MZ^*_{\mathrm{Gn}}(G) = G$ or $MZ^*_{\mathrm{Gn}}(G) = M$. If $MZ^*_{\mathrm{Gn}}(G) = M$, then, clearly, $Z^*_{\mathrm{Gn}}(G) \leqslant M$. On the other hand, if $MZ^*_{\mathrm{Gn}}(G) = G$, then G is supersolvable and so $M = G$. Hence $Z^*_{\mathrm{Gn}}(G) \leqslant M$.

REMARK. We have not yet been able to prove or disprove that $Z^*_{\mathrm{Gn}}(G)$ is the intersection of the maximal supersolvable subgroups of G.

3. **Generalized hypercentral subgroups.** The notion of hypercentral subgroups, introduced by Baer, is extended. It is shown that the generalized hypercenter is the product of all generalized hypercentral subgroups.

DEFINITION. We shall call a normal subgroup H of a group G a generalized hypercentral (GH-central) subgroup of G if for all $M \leqslant H$ and $M \lhd G$, $H/M \cap Z_{\mathrm{Gn}}(G/M) \neq \bar{1}$.

PROPOSITION 3.1. *If H is a GH-central subgroup of G, then for each $N \lhd G$ and $N \leqslant H$, H/N is a GH-central subgroup of G/N.*

LEMMA 3.2. *If $H \leqslant Z_{\mathrm{Gn}}(G)$ and $H \lhd G$, then H is a GH-central subgroup of G. In particular, $Z_{\mathrm{Gn}}(G)$ is a GH-central subgroup of G.*

PROOF. Let $K \lhd G$ and $K \leqslant H$. By (1.3), $Z_{\mathrm{Gn}}(G)K/K = Z_{\mathrm{Gn}}(G)/K \leqslant Z_{\mathrm{Gn}}(G/K)$. Since $H/K \leqslant Z_{\mathrm{Gn}}(G)/K$, $H/K \cap Z_{\mathrm{Gn}}(G/K) = H/K \neq \bar{1}$.

THEOREM 3.3. *For $1 \leqslant i \leqslant m$, every member $(Z_{\mathrm{Gn}}(G))_i$ of the chain*

$$1 = (Z_{\mathrm{Gn}}(G))_0 < Z_{\mathrm{Gn}}(G) = (Z_{\mathrm{Gn}}(G))_1 < (Z_{\mathrm{Gn}}(G))_2 < \cdots < (Z_{\mathrm{Gn}}(G))_m = Z^*_{\mathrm{Gn}}(G)$$

is a GH-central subgroup of G.

PROOF. We use induction on $|G|$. In view of the preceding lemma, we prove the theorem for $i = 2, 3, \ldots, m$. Consider $G/Z_{\mathrm{Gn}}(G)$ and form the chain

$$\bar{1} = (Z_{\mathrm{Gn}}(G))_1/Z_{\mathrm{Gn}}(G) < (Z_{\mathrm{Gn}}(G))_2/Z_{\mathrm{Gn}}(G) < \cdots < (Z_{\mathrm{Gn}}(G))_m/Z_{\mathrm{Gn}}(G) = Z^*_{\mathrm{Gn}}(G)/Z_{\mathrm{Gn}}(G),$$

where

$$Z_{\mathrm{Gn}}(G/Z_{\mathrm{Gn}}(G)/(Z_{\mathrm{Gn}}(G))_{i-1}/Z_{\mathrm{Gn}}(G)) = (Z_{\mathrm{Gn}}(G))_i/Z_{\mathrm{Gn}}(G)/(Z_{\mathrm{Gn}}(G))_{i-1}/Z_{\mathrm{Gn}}(G)$$

for $i = 2, 3, \ldots, m$. Now suppose $M \nleqslant (Z_{\mathrm{Gn}}(G))_i$ and $M \lhd G$, for $i > 1$. If $Z_{\mathrm{Gn}}(G) \leqslant M$, then

$$(Z_{\mathrm{Gn}}(G))_i/Z_{\mathrm{Gn}}(G)/M/Z_{\mathrm{Gn}}(G) \cap Z_{\mathrm{Gn}}(G/Z_{\mathrm{Gn}}(G)/M/Z_{\mathrm{Gn}}(G)) \neq \bar{1}$$

since by induction $(Z_{\mathrm{Gn}}(G))_i/Z_{\mathrm{Gn}}(G)$ is a GH-central subgroup of $G/Z_{\mathrm{Gn}}(G)$. This means that $(Z_{\mathrm{Gn}}(G))_i/M \cap Z_{\mathrm{Gn}}(G/M) \neq \bar{1}$. On the other hand, if $Z_{\mathrm{Gn}}(G) \nleqslant M$, then there is a generalized central element g of G such that $g \notin M$. By (1.3), $gM \in Z_{\mathrm{Gn}}(G/M)$ and so $gM \in (Z_{\mathrm{Gn}}(G))_i/M \cap Z_{\mathrm{Gn}}(G/M)$. Hence $(Z_{\mathrm{Gn}}(G))_i$ is a GH-central subgroup of G.

LEMMA 3.4. *If N_1 and N_2 are GH-central subgroups of a group G, then the product $N_1 N_2$ is also a GH-central subgroup of G.*

PROOF. Let $M \lhd G$ with $M \nleqslant N_1 N_2$. If $M \nleqslant N_1$, or $M \nleqslant N_2$, then $N_1 N_2/M \cap Z_{\mathrm{Gn}}(G/M) \neq \bar{1}$ since $N_i/M \cap Z_{\mathrm{Gn}}(G/M) \neq \bar{1}$ for $i = 1$ or 2. If $M \cap N_1 = N_1$ and $M \cap N_2 = N_2$, then $N_1 N_2 \leqslant M$, which is impossible. Thus we may assume without loss of generality that $M \cap N_1 = W \nleqslant N_1$. Since W is also normal in G, $N_1/W \cap Z_{\mathrm{Gn}}(G/W) \neq \bar{1}$. Let xW be a nonidentity element of $N_1/W \cap Z_{\mathrm{Gn}}(G/W)$. Since G/M is a homomorphic image of G/W and $x \notin M, xM$ is a nonidentity element of $N_1 N_2/M \cap Z_{\mathrm{Gn}}(G/M)$. Hence $N_1 N_2$ is a GH-central subgroup of G.

From the above results we now obtain the following characterization of the generalized hypercenter.

THEOREM 3.5. *$Z^*_{\mathrm{Gn}}(G)$ of a group G is the product of all GH-central subgroups of G.*

PROOF. Lemma 3.4 yields that the product P of all GH-central subgroups of G is GH-central. Hence, $Z^*_{\mathrm{Gn}}(G) \leqslant P$ by Theorem 3.3. If $Z^*_{\mathrm{Gn}}(G) \nleqslant P$, then $P/Z^*_{\mathrm{Gn}}(G) \cap Z_{\mathrm{Gn}}(G/Z^*_{\mathrm{Gn}}(G)) \neq \bar{1}$ and so $Z_{\mathrm{Gn}}(G/Z^*_{\mathrm{Gn}}(G)) \neq \bar{1}$, an impossibility. Thus $Z^*_{\mathrm{Gn}}(G) = P$.

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