LIE *-TRIPLE HOMOMORPHISMS INTO VON NEUMANN ALGEBRAS

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Abstract. Let $M$ and $N$ be associative *-algebras. A Lie *-triple homomorphism of $M$ into $N$ is a *-linear map $\phi: M \to N$ such that

$$\phi[[A, B], C] = [[\phi(A), \phi(B)], \phi(C)].$$

(Here $M$ and $N$ are considered as Lie *-algebras with $[X, Y] = XY - YX$.)

In this note we prove that if $N$ is a von Neumann algebra with no central abelian projections and if $\phi$ is onto, there exists a central projection $D$ in $N$ such that $D\phi$ is a Lie *-homomorphism of $[M, M]$, and $(I - D)\phi$ is a Lie *-antihomomorphism of $[M, M]$.

1. Introduction. An associative algebra $M$ can be turned into a Lie algebra by defining a new multiplication $[X, Y] = XY - YX$ where $XY$ is the associative product of $X$ and $Y$. Every abstract Lie algebra is isomorphic to a subalgebra of a Lie algebra formed in this way. A Lie triple system is a subspace of $M$ closed under the Lie triple product $[[A, B], C]$. Lie triple systems and their homomorphisms have been studied in relation to Jordan homomorphisms of rings and the following theorem proved [2, Theorem 15]:

Let $\phi$ be a Lie triple system homomorphism of the special Lie ring $L$ and denote by $M$ the enveloping Lie ring of $\phi(L)$ and $Z$ the centre of $M$. Assume (i) $M/Z$ has no commutative Lie ideals and (ii) any two nonzero Lie ideals in $M/Z$ have nonzero intersection. Then $\phi$, when restricted to the Lie ring $[L, L]$, is either a Lie homomorphism or antihomomorphism.

We wish to prove an analogous theorem when the image algebra is a von Neumann algebra. The situation is complicated by the presence, in the general case, of nonzero central projections which makes (ii) of the above theorem inapplicable.

2. Notation and preliminaries. $M$ is a *-algebra over the complex field and $M_0, M_1$ subsets of $M$, then $[M_0, M_1] = \{[A, B] : A \in M_0, B \in M_1\}$ is all finite linear combinations of elements of the form $[A, B]$ with $A \in M_0, B \in M_1$. A Lie *-triple homomorphism $\phi: M \to N$ is a *-linear map preserving the Lie triple product $[[A, B], C]$. The enveloping Lie algebra [2, p. 493] of $\phi(M)$ is the set $\phi(M) + [\phi(M), \phi(M)]$. A Lie *-ideal of $M$ is a *-linear subspace $U \subseteq M$ such that if $Y \in U$, $[X, Y] \in U$ for all $X \in M$.

A von Neumann algebra $M$ is a weakly closed, selfadjoint algebra of operators on a complex Hilbert space $H$ containing the identity operator $I$. The set $Z_M = \{S \in M : [S, T] = 0 \text{ for all } T \in M\}$ is called the centre of $M$. 

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If $P$ is a projection (= selfadjoint idempotent) in $M$, then $M_P = \{PAP | A \in M\}$. A projection $P$ is abelian if $M_P$ is an abelian algebra. We use [1] as a general reference for the theory of von Neumann algebras.

The following fact will be used several times in what follows: If $M$ is a C*-algebra, $X, Y \in M$ with $Y = Y^*$, then $[X, Y] \in Z_M$ implies $[X, Y] = 0$ [2, Lemma 6]. This implies, for example, that if $M_0$ and $M_1$ are subsets of $M$ with $M_1$ a *-subspace, then $[M_0, M_1] \subseteq Z_M$ implies $[M_0, M_1] = \{0\}$.

3. Lie *-triple homomorphisms. Let $\phi : M \to N$ be a Lie *-triple homomorphism where $M$ is a *-algebra over $C$ and $N$ is a von Neumann algebra. The case where $N$ is a factor (that is $Z_N = \{\lambda I : \lambda \in C\}$) is included separately, even though the factor case fits into the general theorem, since $\phi$ can be analyzed when $N$ is a factor by using the Jacobson-Rickart theorem already mentioned. The following result may be of independent interest.

**Lemma 1.** Let $N$ be a C*-algebra. Then $N/Z_N$, considered as a Lie *-algebra, contains no nontrivial abelian Lie *-ideals.

**Proof.** Let $N_0$ be an abelian Lie *-ideal in $N/Z_N$ and let $\pi : N \to N/Z_N$ be the canonical Lie *-homomorphism where $\pi(A) = A + Z_N$. $N_0$ is generated, as a *-linear space, by selfadjoint elements so let $A + Z_N, B + Z_N$ be elements of $N_0$ with $A - A^* \in Z_N$ and $B - B^* \in Z_N$. Then $\pi([A, B]) = [\pi(A), \pi(B)] = 0$ since $N_0$ is abelian. Thus $[A, B] \in \ker \pi = Z_N$. Now $A - A^* \in Z_N$ implies $[A^*, B] = [A, B] \in Z_N$ so that $[A + A^*, B] \in Z_N$. This forces $[A + A^*, B] = 0$. Similarly $[A - A^*, B] = 0$. Adding, we have $[A, B] = 0$. $\pi^{-1}(N_0) + Z_N$ is therefore an abelian Lie *-ideal in $N$ so that by [3, Lemma 36], $\pi^{-1}(N_0) \subseteq Z_N$ or $N_0 = \{0\}$.

**Theorem.** If $N$ is a factor and $\phi : M \to N$ is a Lie *-triple homomorphism of $M$ onto $N$ then $\phi([M, M])$ is a Lie *-homomorphism or a Lie *-antihomomorphism.

**Proof.** Since $\phi$ is onto, $\phi(M) + [\phi(M), \phi(M)] = N$ so that we need only show condition (ii) of the Jacobson-Rickart theorem is fulfilled. Let $U_1$ and $U_2$ be nonzero Lie *-ideals in $N/Z_N$ and let $V_1 = \pi^{-1}(U_1)$, $V_2 = \pi^{-1}(U_2)$. Then $V_1 + Z_N, V_2 + Z_N$ are Lie *-ideals in $N$ and neither is contained in $Z_N$.

By [3, Lemma 37] there exist nonzero two-sided ideals $\mathcal{J}_1, \mathcal{J}_2$ of $N$ such that $[\mathcal{J}_1, N] \subseteq V_1 + Z_N$ and $[\mathcal{J}_2, N] \subseteq V_2 + Z_N$. If $[\mathcal{J}_1, N] \subseteq Z_N$ then $[\mathcal{J}_1, N] = 0$ and $\mathcal{J}_1 \subseteq Z_N = \{\lambda I : \lambda \in C\}$ which would force $\mathcal{J}_1 = \{0\}$. If $\mathcal{J}_1 = N$ then

$$[\mathcal{J}_1, N] \cap [\mathcal{J}_2, N] = [\mathcal{J}_2, N] \subseteq (V_1 + Z_N) \cap (V_2 + Z_N)$$

so that $U_1 \cap U_2 \neq \{0\}$.

So we can assume $U_1 \cap U_2 = \{0\}$ and $\mathcal{J}_1, \mathcal{J}_2$ are nonzero, proper ideals in $N$. Now

$$\pi^{-1}(\{0\}) = \pi^{-1}(U_1 \cup U_2) = (V_1 + Z_N) \cap (V_2 + Z_N) \subseteq Z_N.$$ 

Hence $[\mathcal{J}_1, N] \cap [\mathcal{J}_2, N] \subseteq Z_N$ which implies $[\mathcal{J}_1, \mathcal{J}_2] \subseteq Z_N$. Since $V_1 + Z_N$ and $V_2 + Z_N$ are selfadjoint collections, we can assume the same of $\mathcal{J}_1$ and $\mathcal{J}_2$ so that $[\mathcal{J}_1, \mathcal{J}_2] = \{0\}$. Moreover $[\mathcal{J}_1, \mathcal{J}_2, N] \subseteq [\mathcal{J}_1, N] \cap [\mathcal{J}_2, N] \subseteq Z_N$ so that $[\mathcal{J}_1, \mathcal{J}_2, N] = \{0\}$. Hence $\mathcal{J}_1, \mathcal{J}_2$ is a selfadjoint two-sided ideal in $Z_N$ so
that \( \mathcal{I}_1 \mathcal{I}_2 = \{0\} \) which is impossible since \( N \) is a factor.

We now turn our attention to the general case. As in [2] the sets

\[
N^+ = \left\{ \sum_{i=1}^{n} \phi[A_i, B_i] - [\phi(A_i), \phi(B_i)]: A_i, B_i \in M \right\}
\]

and

\[
N^- = \left\{ \sum_{i=1}^{n} \phi[A_i, B_i] - [\phi(B_i), \phi(A_i)]: A_i, B_i \in M \right\}
\]

are Lie ideals in \( \phi(M) + [\phi(M), \phi(M)] \). In our case \( N^+ \) and \( N^- \) are also closed under the \(*\)-operation since \( \phi \) preserves adjoints. If, for example, \( N^+ \subseteq Z_N \) then

\[
0 = [\phi[A, B] - [\phi(A), \phi(B)], \phi[X, Y]] = [\phi[A, B], \phi[X, Y]] - [\phi[A, B], [X, Y]]
\]

so that \( \phi \) is a Lie \(*\)-homomorphism of \([M, M] \). Similarly, if \( N^- \subseteq Z_N \) then \( \phi \) is a Lie \(*\)-antihomomorphism of \([M, M] \).

**Lemma 2.** Let \( \phi: M \to N \) be a Lie \(*\)-triple homomorphism of the \(*\)-algebra \( M \) onto a von Neumann algebra \( N \) which has no abelian central projections and suppose \( N^+ \not\subseteq Z_N \) and \( N^- \not\subseteq Z_N \). There exist projections \( C \neq 0 \) and \( D \neq 0 \) in \( Z_N \) such that \( N^+ + Z_N \subseteq N_C + Z_N \), \( N^- + Z_N \subseteq N_D + Z_N \) and \( CD = 0 \).

**Proof.** By [2, Theorem 14] we have \([N^+, N^-] \subseteq Z_N \) and so \([N^+, N^-] = 0 \), since \( N^+, N^- \) are selfadjoint collections. Hence \( N^+, N^- \) are commuting Lie \(*\)-ideals so that \((N^+ + Z_N)^{-\text{uw}} \) and \((N^- + Z_N)^{-\text{uw}} \) are also commuting Lie \(*\)-ideals. \((N^+ + Z_N)^{-\text{uw}} \) is the ultra-weak closure of \((N^+ + Z_N) \). By [3, Theorem 4, Corollary], \((N^+ + Z_N)^{-\text{uw}} = N_C + Z_N \), \((N^- + Z_N)^{-\text{uw}} = N_D + Z_N \) where \( C \neq 0 \), \( D \neq 0 \) are projections in \( Z_N \). Since these Lie \(*\)-ideals commute we have \([N_C, N_D] = [N_{CD}, N_{CD} = 0 \) or \( CD = 0 \) is a central abelian projection. Thus \( CD = 0 \).

**Theorem 2.** Let \( \phi: M \to N \) be a Lie \(*\)-triple homomorphism of a \(*\)-algebra \( M \) onto a von Neumann algebra \( N \) which has no central abelian projections. There exists a projection \( D \in Z_N \) such that \( D\phi \) is a Lie \(*\)-homomorphism on \([M, M] \) and \((I - D)\phi \) is a Lie \(*\)-antihomomorphism on \([M, M] \).

**Proof.** If \( N^+ \subseteq Z_N \) or \( N^- \subseteq Z_N \) then \( D = 0 \) or \( D = I \). Otherwise there exist projections \( C \neq 0 \), \( D \neq 0 \) in \( Z_N \) such that \( N^+ + Z_N \subseteq N_C + Z_N \), \( N^- + Z_N \subseteq N_D + Z_N \) and \( CD = 0 \). We have \( N^+D = \{TD \mid T \in N^+ \} \subseteq Z_ND \) and \( N^-C \subseteq Z_NC \). By the discussion before Lemma 2 we have that \( D\phi \) is a Lie \(*\)-homomorphism of \([M, M] \) and \( C\phi \) is a Lie \(*\)-antihomomorphism of \([M, M] \).

Now \( N^+(I - C - D) \subseteq Z_N(I - C - D) \) and \( N^-(I - C - D) \subseteq Z_N(I - C - D) \) so that \( (I - C - D)\phi \) is both a Lie \(*\)-homomorphism and a Lie \(*\)-antihomomorphism on \([M, M] \). Thus if \( X, Y \in [M, M] \),

\[
(I - C - D)\phi[X, Y] = (I - C - D)[\phi(X), \phi(Y)] = (I - C - D)[\phi(Y), \phi(X)].
\]
This implies \((I - C - D) \phi (X) \phi (Y) = (I - C - D) \phi (Y) \phi (X)\) or that \((I - C - D) \phi [M, M]\) is abelian. \([M, M], M) \subseteq [M, M]\) so that
\[
\left[ \left[ \phi(M), \phi(M) \right], \phi(M) \right] \subseteq \phi[M, M].
\]
Since \(\phi\) is onto, \([N, N], N) \subseteq \phi[M, M]\) and
\[
\left[ \left[ N_{(I-C-D)}, N_{(I-C-D)} \right], N_{(I-C-D)} \right] \subseteq (I - C - D)\phi[M, M].
\]
Hence \([N_{(I-C-D)}, N_{(I-C-D)}, N_{(I-C-D)}] \subseteq (I - C - D)\phi[M, M].\)

**Remark 1.** The requirement that \(\phi\) be onto is made so that \(N^+\) and \(N^-\), which are Lie \(*\)-ideals of \(\phi(M) + [\phi(M), \phi(M)]\) will be Lie \(*\)-ideals in \(N\) where a characterization of such ideals is known. Other restrictions on \(M, N\) and \(\phi\) can be made to insure that \(\phi(M) + [\phi(M), \phi(M)] = N\). \(\phi\) is called \(L\)-onto if, given \(Y \in N\), there exists \(X \in M\) such that \(\phi(X) - Y \in Z_N\).

If \(N\) is an infinite von Neumann algebra then 
\([N, N] = N [5, \text{Theorem 2}]\). Hence if \(\phi\) is \(L\)-onto and \(N\) is infinite, \([\phi(M), \phi(M)] = [N, N] = N\). If \(N\) is a type I finite von Neumann algebra then \(Z_N + [N, N] = N [4, \text{Theorem 1}]\). If in this case \(\phi\) were \(L\)-onto and \(Z_N \subseteq \phi(M)\), we would have \(N = Z_N + [N, N] \subseteq \phi(M) + [\phi(M), \phi(M)] \subseteq N\).

**Remark 2.** Modification of the arguments of [3] shows that if \(M\) and \(N\) are von Neumann algebras with no central abelian projections and \(\phi\) is \(L\)-onto, then \(Z_M\) and \(Z_N\) are \(*\)-isomorphic.

**References**


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