LOCAL RESOLVENTS OF OPERATORS WITH ONE-DIMENSIONAL SELF-COMMUTATOR

CONSTANTIN APOSTOL AND KEVIN CLANCEY

ABSTRACT. Let $T = H + iJ$ be an irreducible operator on a Hilbert space with one-dimensional self-commutator. It is known that the selfadjoint operator $H$ is absolutely continuous. Let $E_H$ denote the absolutely continuous support of $H$. In this note the following theorem is proven:

**Theorem.** If there exists a real number $p$ such that $\text{ess inf } E_H < p < \text{ess sup } E_H$, then the operator $T$ has a nontrivial invariant subspace.

Let $\mathcal{H}$ be a separable Hilbert space with inner product $(\cdot, \cdot)$. Let $T$ be a bounded linear operator on $\mathcal{H}$. The operator $T$ is called hyponormal in case its self-commutator $T^*T - TT^* = D$ is nonnegative. If the adjoint $T^*$ is a hyponormal operator, then $T$ is called cohyponormal. If the self-commutator $T^*T - TT^*$ is a one-dimensional operator, then either $T$ or $T^*$ is hyponormal.

It is not known at present whether every operator with a one-dimensional self-commutator has a nontrivial invariant subspace. In this note a result is described which increases the class of operators with one-dimensional self-commutator that are known to have nontrivial invariant subspaces.

Let $T$ be a cohyponormal operator. The cartesian decomposition of $T$ will be expressed $T = H + iJ$, where $H$ and $J$ are selfadjoint. For the purposes of this note it can be assumed that $T$ is an irreducible operator. In this case the selfadjoint operators $H$ and $J$ are absolutely continuous [10, Theorem 3.2.1]. Suppose $H = \int t\,dG_t$ is the spectral resolution of $H$. Then there is a Borel set $E_H$ in the real line, determined up to a set of measure zero with the property $\int_{E_H} dG_t = I$ and if $\int_F dG_t = I$, for $F \subset E_H$, then $E_H \setminus F$ is of Lebesgue measure zero. This set $E_H$ (really $E_H$ is an equivalence class of Borel sets) will be called the absolutely continuous support of $H$.

It is known [10, Theorem 3.4.1] that if $T = H + iJ$ is cohyponormal, then $\sigma(H)$ (the spectrum of $H$) is the projection on the $x$-axis of $\sigma(T)$. Similarly, $\sigma(J)$ is the projection of $\sigma(T)$ onto the $y$-axis. An operator with disconnected spectrum is known to have nontrivial invariant subspaces. It follows that every cohyponormal operator $T = H + iJ$, where $\sigma(H)$ or $\sigma(J)$ is not an interval, has a nontrivial invariant subspace.

The main result in this note is the following:

**Theorem 1.** Let $T = H + iJ$ be an irreducible operator with one-dimensional self-commutator. Let $E_H$ be the absolutely continuous support of $H$ and assume...
\( \sigma(H) = [a, b] \). If for some \( p \in (a, b) \), \( \int_{E_H} |t - p|^{-1} dt < \infty \), then the operator \( T \) has a nontrivial invariant subspace.

The proof of Theorem 1 will be based on some estimates of the local resolvent of the vector in the range of \( T^* T - TT^* \). These estimates are deduced from results in Carey and Pincus [1]. The fact that local resolvents can be used to obtain invariant subspaces for hyponormal operators is in Stampfli [12]. In the case where \( T \) is cohyponormal with no point spectrum \( TT^* - T^* T = \langle \varphi, \varphi \rangle \), then there is a weakly continuous extension of \( (T - \lambda)^{-1} \varphi \) onto the whole complex plane \( \mathbb{C} \) (see Putnam [11]). In the final section of this note it is shown that this extension of \( (T - \lambda)^{-1} \varphi \) is not an analytic extension across the boundary of \( \sigma(T) \).

1. Preliminaries and notations. From now on it will be assumed that \( T = H + iJ \) is an irreducible cohyponormal operator with one-dimensional self-commutator. The notation \( \mathbb{T}_\lambda \) will be used for \( T - \lambda I \). It is convenient to normalize \( T \) such that \( TT^* - T^* T = \langle \phi, \phi \rangle \), where \( \phi \) is a unit vector in \( \mathcal{H} \).

J. D. Pincus [7] has provided a useful unitary invariant for the operator \( T \). There is a bounded measurable function \( g \) defined on the plane which satisfies

\[
1 + (\pi i)^{-1} ((J - \omega)^{-1} (H - z)^{-1} \varphi, \varphi) = \exp \left\{ \frac{1}{2\pi i} \int \int g(x + iy) \frac{dx\,dy}{(x - z)(y - \omega)} \right\},
\]

where \( \text{Im } z \neq 0 \) and \( \text{Im } \omega \neq 0 \). The function \( g \) is called the principal function of \( T \) and has the following properties:

(i) The function \( g \) vanishes off \( E_H \times E_J \) and satisfies \( 0 \leq g \leq 1 \) (see [1]).

(ii) The spectrum of \( T \) consists of all complex \( \lambda \) such that every neighborhood of \( \lambda \) intersects the set \( \{ z : g(z) \neq 0 \} \) in a set of positive measure (see [4] and [8]).

(iii) The essential spectrum of \( T \) consists of all complex \( \lambda \) such that every neighborhood of \( \lambda \) intersects the sets \( \{ z : g(z) \neq 0 \} \) and \( \{ z : g(z) \neq 1 \} \) in a set of positive measure [1].

(iv) The point spectrum of \( T \) consists of all complex \( \lambda = \mu + iv \) such that

\[
\mathcal{B}(\lambda) = \{ x : |x - \lambda| < 1 \}, [1].
\]

Let \( \lambda = \mu + iv \) be a complex number. It is useful to introduce the function

\[
\nu_g(\lambda) = \frac{1}{\pi} \int \int \frac{g(x + iy)}{(x - \mu)^2 + (y - v)^2} dx\,dy.
\]

For our purposes we will need the following estimate which follows easily from the results in [1], (see, in particular the proof of Theorem 6 of [1]).

(v) Let \( \lambda \) be complex and let \( \mathbb{T}_\lambda^* \mathbb{T}_\lambda = \int i\,dE_n, \mathbb{T}_\lambda \mathbb{T}_\lambda^* = \int i\,dF_t \) be the spectral resolutions of \( \mathbb{T}_\lambda^* \mathbb{T}_\lambda \) and \( \mathbb{T}_\lambda \mathbb{T}_\lambda^* \), respectively. Then
\[ 1 + \frac{2}{\pi} \int t^{-1} d(E_t \varphi, \varphi) = \exp[\nu_g(\lambda)], \]
\[ 1 - \frac{2}{\pi} \int t^{-1} d(F_t \varphi, \varphi) = \exp[-\nu_g(\lambda)]. \]

In case the operator \( H \) has simple spectrum, then Dao-Xeng Xa [13] has shown that \( T \) is unitarily equivalent to an operator \( S \) acting on \( L^2(E_H) \) of the form:

\[ \text{(2)} \quad Sf(s) = sf(s) + i \left[a(s)f(s) + \frac{b(s)}{\pi i} \int_{E_H}^* \frac{b(t)f(t)}{i - s} dt \right]; \]

where, \( a, b \) are bounded measurable functions on \( E_H \), \( b(t) \neq 0 \) a.e. and \( a \) is real valued. The singular integral is interpreted as a Cauchy principal value, that is, \( \int^* = \lim_{\epsilon \to 0} \int_{|t-s|>\epsilon} \) vector valued generalization of the above result of Dao-Xeng Xa is in [7]. In fact, the theory of unitary invariants for operators with trace class self-commutator has been significantly developed [3], [6].

In order to have concrete examples of the determining function we observe that the determining function for the operator \( S \) defined by (2) is the function

\[ g(x + iy) = \frac{\arg}{\pi} \left[ \frac{a(x) + |b(x)|^2 - y}{a(x) - |b(x)|^2 - y} \right]; \]

where, \( 0 \leq \arg \leq 2\pi \).

Let \( A \) be an operator on \( \mathcal{H} \) which has no point spectrum. For \( x \in \mathcal{H} \), the function \( x(\lambda) = (A - \lambda)^{-1}x \) has a maximal \( \mathcal{H} \)-valued analytic continuation from the resolvent set of \( A \). The domain of this maximal analytic continuation is called the local (analytic) resolvent set of \( x \) and is denoted by \( \rho_A(x) \). The complementary set is called the local spectrum of \( x \) and will be denoted by \( \sigma_A(x) \). If \( \sigma \) is a compact subset in the plane, then it is easy to see that \( M(\sigma) = \{ x \in \mathcal{H}: \sigma_A(x) \subset \sigma \} \) is a linear manifold in \( \mathcal{H} \) which is invariant under the operator \( A \). Stampfli [12] has shown that \( M(\sigma) \) is sometimes a closed subspace when \( A \) is hyponormal. More precisely:

**Theorem 2.** Let \( A \) be a hyponormal operator such that \( A^* \) has no point spectrum. If \( \sigma \) is a compact subset of the plane, then \( M(\sigma) \) is a closed subspace invariant under the operator \( A \).

Unfortunately, there are hyponormal operators such that \( M(\sigma) = \mathcal{H} \) or \( M(\sigma) = \{0\} \) for every compact set \( \sigma \). One such example is the unilateral shift. On the other hand there are plenty of examples where the subspace \( M(\sigma) \) is a nontrivial invariant subspace for a hyponormal operator (see, e.g. [9] and [12]). Our proof of Theorem 1 will provide more examples where \( M(\sigma) \) is nontrivial.

It is interesting to compare the problem of local analytic spectra in the hyponormal case with the cohyponormal case. If \( A \) is a cohyponormal operator with empty point spectrum, then there exists a vector \( x \in \mathcal{H} \) such that \( \sigma_A(x) \) is a nonempty proper subset of \( \sigma(A) \) [12], however, no analogue of Theorem 2 is known in the cohyponormal case.

If \( x \) is an operator on \( \mathcal{H} \) which is one-to-one and \( AA^* = \int t \, dG_t \) is the spectral resolution of \( AA^* \), then the vector \( x \in \mathcal{H} \) belongs to the domain of
A^{-1}$ if and only if $\int t^{-1} d(G_t x, x) < \infty$. In this case the norm of $A^{-1} x$ verifies $\|A^{-1} x\|^2 = \int t^{-1} d(G_t x, x)$. Property (v) describes when $\varphi$ is in the domain of $T^{-1}$ or $((T^*)^*)^{-1}$.

An irreducible hyponormal operator has no point spectrum. It follows that $\varphi$ is in the domain of $((T^*)^*)^{-1}$ if and only if $\nu_g(\lambda) < \infty$. It should be observed that $\nu_g(\lambda) = \infty$ a.e. on the set $\{z: g(z) \neq 0\}$. The fact that the characteristic function of $E_H \times E_J$ dominates the function $g$ implies that the condition

$$\int_{E_H} |t - p|^{-1} dt < \infty$$

is sufficient for $\varphi$ to be in the domain of $((T^*)^*)^{-1}$, for all $\lambda$ such that $\Re \lambda = p$. In fact condition (3) implies $\|((T^*)^*)^{-1} \varphi\|$ is bounded on $\Re \lambda = p$.

Let $E$ be a bounded measurable set. Define for real $p$ the function $v_E(p) = \int_E |t - p|^{-1} dt$. This function $v_E$ is infinite at a.e. point in $E$. On the other hand $v_E(p)$ is often finite for points $p \notin E$, for example, if $E$ is closed, then $v_E(p) < \infty$, for $p \notin E$.

There are a couple of examples which are interesting in connection with the function $v_E$. There is a set $E_L \subset I = [0, 1]$ of positive measure such that $I \setminus E_L$ is a closed nowhere dense set of positive measure and $v_{E_L}(p) = \infty$, for all $p \in I \setminus E_L$. This last example is due to Lusin (see, e.g. [2]). On the other hand, there is another set $E_0$ of positive measure in $I$ such that $I \setminus E_0$ is a closed nowhere dense set of positive measure with $v_{E_0}(p) < \infty$, for a.e. $p \in E_0$ (see [2]).

The second equality in property (v) implies that $\varphi$ is in the domain of $T^{-1}$, whenever, $\lambda$ is not in the point spectrum of $T$; moreover, $\|T^{-1} \varphi\| \leq 1$. This last result has a generalization in Putnam [11].

2. Proof of Theorem 1. We can assume that both operators $T$ and $T^*$ have empty point spectrum. It is clear from the discussion in the preceding section that condition (3) implies $((T^*)^*)^{-1} \varphi$ exists as a bounded $\mathbb{K}$-valued function on $\Re \lambda = p$. An argument similar to that given in Putnam [11] can be used to show that the function $\varphi(\lambda) = (T^* - \lambda)^{-1} \varphi$ is weakly continuous on $\Re \lambda = p$.

Let $C$ be a circle that contains $\sigma(T^*)$ in its interior and let $\gamma$ be the open subinterval of $\Re \lambda = p$ which is contained inside the circle $C$. Form the two closed contours $\Gamma_1 = \gamma \cup [C \cap [\Re \lambda \geq p]]$ and $\Gamma_2 = \gamma \cup [C \cap [\Re \lambda < p]]$. We assume $\Gamma_1, \Gamma_2$ are oriented in the counter-clockwise direction. One of the vectors $\varphi_i = \int_{\Gamma_i} \varphi(\lambda) d\lambda, i = 1, 2$, is nonzero; here, the integral is defined as a weak integral.

For convenience we assume $\varphi_1 \neq 0$. It is routine to prove that $\varphi_1(\mu) = \int_{\Gamma_1} \varphi(\lambda)(\lambda - \mu)^{-1} d\lambda$ is an analytic continuation of $\varphi_1(\mu) = (T^* - \mu)^{-1} \varphi_1$ onto the domain exterior to $\Gamma_1$. This means that $\sigma_{T^*}(\varphi_1)$ is a proper subset of $\sigma(T^*)$. It follows from Theorem 2 that $M(\sigma_{T^*}(\varphi_1))$ is a nontrivial subspace invariant under $T^*$. This completes the proof.

3. Conclusion. It is interesting to compare the result in Theorem 1 with the invariant subspace result which follows from property (iv) of the determining function. For definiteness we work with a specific example. Let $E$ be a bounded measurable subset of the real line and consider the singular integral
operator $T_E$ defined on $L^2(E)$ by

$$T_E f(s) = sf(s) + \frac{1}{\pi} \int_E \frac{f(t)}{t-s} dt.$$  

Of course, $T_E$ is one of the cohyponormal operators defined in equation (2). The principal characteristic function of $E \times [-1,1]$.

The operator $T_E$ has an eigenvalue if and only if for some real $p$, the function $\nu_E(p) < \infty$; here, $E^c$ is the complement of $E$ relative to some bounded interval containing $E$ in its interior. Theorem 1 gives a nontrivial invariant subspace of $T_E$ when $\nu_E(p) < \infty$, for some $p$ satisfying $\text{ess inf } E < p < \text{ess sup } E$.

The proof of Theorem 1 shows that the operator $T_E$ has plenty of nontrivial invariant subspaces whenever $F \subseteq E_L$. It does not give a similar statement about $T_{E_L}(\nu_{E_L}(p) = \infty$ on $(0,1))$, however, $T_{E_L}$ has plenty of eigenvalues.

It was shown above that in the absence of point spectrum of $T$, then $\varphi(\lambda) = (T - \lambda)^{-1} \varphi$ admits a weakly continuous extension (still denoted by $\varphi(\lambda)$) from the resolvent set onto all of $C$. It develops that $\varphi(\lambda)$ is not analytic at points in the boundary of $\sigma(T)$.

From the fact that $\nu_{E_L}(\lambda) = \infty$ a.e. on $\{z: g(z) \neq 0\}$, it follows that $\|\varphi(\lambda)\| = 1$ for points in every neighborhood of a point $\lambda_0$ in the boundary of $\sigma(T)$. On the other hand $\|\varphi(\lambda)\| < 1$, for $\lambda \notin \sigma(T)$. This implies that $\varphi(\lambda)$ is not analytic in any neighborhood of $\lambda_0$ (see, e.g. Dunford and Schwartz [5, p. 220]).

REFERENCES

3. ———, Commutators, symbols and determining functions (preprint).