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ON RELATIVELY FREE SUBSETS OF LIE GROUPS

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ABSTRACT. In an arbitrary neighborhood U of the identity e of a connected Lie group there is a subset S of cardinality $\mathfrak{c}$ and *relatively free*, i.e., the only nontrivial equations $x_1^{\varepsilon_1}x_2^{\varepsilon_2}\cdots x_n^{\varepsilon_n} = e$, $\varepsilon_i = \pm 1$, satisfied by substitution for distinct symbols among the x_i distinct elements of S are equations that are identities throughout G.

0. In [2] the existence of *the* free topological group $F(X)$ associated with a completely regular space X is shown by a construction involving quaternions. In brief (and corrected) form, the argument proceeds as follows: the (algebraic) free group $F_0(X)$ generated by X is embedded isomorphically in the group $\mathbf{H}_{1\infty}(X) \equiv P_f\mathbf{H}_{1f}$, the Cartesian product of the multiplicative group $\mathbf{H}_1$ of quaternions of norm 1 (each $\mathbf{H}_{1f} = \mathbf{H}_1$) where the index f ranges over $C(X, \mathbf{H}_1)$, the set of continuous maps $f: X \to \mathbf{H}_1$. In this embedding, X preserves its topology and $F_0(X)$ is endowed with the topology of a topological group. Standard results show that $\mathfrak{T}_{\max} = \sup\{\mathfrak{T}: F_0(X)$ is a topological group in the topology $\mathfrak{T}$ and X inherits its topology from $\mathfrak{T}\}$ $\equiv \sup\{TGF_0(X)\}$ produces $F(X)$. The role played by the compact group $\mathbf{H}_{1\infty}(X)$ is that of insuring that the set $TGF_0(X)$ is nonempty.

The argument hinges on the existence of an infinite free subset of $\mathbf{H}_1$. Since $\mathbf{H}_1$ is a connected Lie group, a question related to this aspect of $\mathbf{H}_1$ is explored below and is answered as follows: In every connected Lie group G there is a subset S, of cardinality $\mathfrak{c} \equiv \operatorname{card}(\mathbf{R})$ and as free as any subset of G can be: if a word evaluated on S yields e (the identity of G), then the word is identically e on G (see below for details). In particular, if no (word) identities hold universally in G, then S is free.

1. The proof depends on two lemmas, the second of which emerged in the form given below as a result of illuminating conversations with Professor S. Schanuel of SUNY/Buffalo. The first lemma is given in [1], [4]. The proof below is somewhat more elementary than that in [4] and is therefore included here.

LEMMA 1. *Let $f(x_1, x_2, \ldots, x_n)$ be a real- or complex-valued function of the real variables $x_1, x_2, \ldots, x_n$. If, for some constant a, the n-dimensional measure $\mu_n(f^{-1}(a))$ is positive, then $f \equiv a$ in any region R where f is analytic and such that $R \supset f^{-1}(a)$.*

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PROOF. We may assume that for some cube $C = \{(x_1, x_2, \ldots, x_n): |x_i - b_i| \leqslant \varepsilon_i > 0, i = 1, 2, \ldots, n\}$ contained in R, $\mu_n(C \cap f^{-1}(a)) > 0$ and that in C

$$f(x_1, \ldots, x_n) = \sum_{0 < k_i} a_{k_1 \cdots k_n}(x_1 - b_1)^{k_1} \cdots (x_n - b_n)^{k_n}$$

converges uniformly and absolutely. Thus we may write

$$f(x_1, \ldots, x_n) = \sum_{0 < k_1} f_{1k_1}(x_2, \ldots, x_n)(x_1 - b_1)^{k_1}$$

where f_{1k_1} are analytic functions of $x_2, \ldots, x_n$. The Fubini theorem shows that for each $(x_2, \ldots, x_n)$ in some set S_{n-1} of positive $(n-1)$-dimensional measure, $\mu_1\{x_1:(x_1, x_2, \ldots, x_n) \in C \cap f^{-1}(a)\} > 0$. The set of such x_1 has a nonempty derived set in $\mathbf{C}$ and so $f_{1k_1}(x_2, \ldots, x_n) \equiv 0$, $k_1 \geqslant 1$, $f_{10}(x_2, \ldots, x_n) = a$ on S_{n-1}.

Since the lemma obtains (in even stronger form!) for $n = 1$, an argument by induction completes the proof.

LEMMA 2. (S. SCHANUEL) *Let ψ: $M \to N$ be an analytic map between analytic manifolds M and N. Let $K = \{x$: $x \in M$, ψ is constant in some neighborhood of $x\}$. Then K is closed (and clearly open).*

PROOF. Let $p_0 \in \overline{K} \equiv$ closure of K in M. Then there are points $p_n \in K$, $n \geqslant 1$, $p_n \to p_0$, and for each $n \geqslant 1$, ψ is constant on some neighborhood V_n of p_n. Let B_M be a cubical (open) neighborhood in $\mathbf{R}^{\dim M}$ and let f_M: $B_M \to M$ provide a chart around p_0; let B_N, f_N: $B_N \to N$ serve similarly for $\psi(p_0)$. Then if the coordinate maps for an f^{-1} are denoted by f_i^{-1} we find

$$h_i \equiv f_{Ni}^{-1} \circ \psi \circ f_M: B_M \to \mathbf{R}, \qquad i = 1, 2, \ldots, \dim M,$$

are analytic maps of B_M into $\mathbf{R}$ and for all $n \geqslant$ some n_1, each is constant on the open subset $f_M^{-1}(f_M(B_M) \cap V_n)$ of B_M. Since $\mu(f_M^{-1}(f_M(B_M) \cap V_n)) > 0$, Lemma 1 shows that each h_i is constant on B_M. Hence $\psi \circ f_M$ is constant on B_M, ψ is constant on $f_M(B_M)$, an open set containing p_0. Hence $p_0 \in K$ and K is closed.

COROLLARY. *If M is connected, $K = \varnothing$ or $K = M$, i.e., a nonconstant analytic map of a connected manifold cannot be locally constant.*

REMARK. There is a clear connection between the results above and those in [1]. The author is obliged to the referee for the citation.

2. We make some clarifying remarks about *words* and groups. A *word* W_{lk} is defined by a finite sequence of l distinct symbols $u_1, u_2, \ldots, u_l$, a finite sequence of signs $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k = \pm 1$, where $k \geqslant l$ and a "product" $w_1^{\varepsilon_1} \cdots w_k^{\varepsilon_k}$ where each w_i is some u_j. It is assumed that in $w_1^{\varepsilon_1} \cdots w_k^{\varepsilon_k}$ there is no finite sequence $w_p, w_{p+1}, \ldots, w_{p+q}$ and some symbol u_j such that $w_p = w_{p+1} = \cdots = w_{p+q} = u_j$ and $\sum_{i=p}^{p+q} \varepsilon_i = 0$. Such situations are inessential and will be excluded from discussion. Thus we deal only with *essential* words.

For a group G and a subset $A \subset G$, the notation $A^{\times l}$ signifies the l-fold Cartesian product of A with itself, as distinguished from A^l, the set $\{a_1 a_2 \cdots a_l: a_i \in A, i = 1, 2, \ldots, l\}$. For a word W_{lk} as described there is defined on $G^{\times l}$ a function $\tilde{W}_{lk}$: $G^{\times l} \to G$ given by

$$\tilde{W}_{lk}(g_1, g_2, \ldots, g_l) = \begin{array}{l}\text{the result of replacing in the "product"} \\ w_1^{\varepsilon_1} \cdots w_k^{\varepsilon_k} \text{ each } u_i \text{ by } g_i,\ i = 1, \\ 2, \ldots, l, \text{ and thereupon calculating} \\ \text{the resulting element of } G.\end{array}$$

A word W_{lk} is *nontrivial* for the group G if $\tilde{W}_{lk} \not\equiv e$ on $G^{\times l}$, where e is the identity of G.

In $G^{\times l}$ there is the "antidiagonal" $E_l = \{(g_1, g_2, \ldots, g_l)\colon g_p \neq g_q$ for $1 \leqslant p \neq q \leqslant l\}$. A subset S of G is called *relatively free* if for every word W_{lk} nontrivial for G, $\tilde{W}_{lk} \neq e$ throughout $S^{\times l} \cap E_l$, $l \leqslant k = 2, 3, \cdots$. If *every* (essential) word W_{lk} is nontrivial for G, then a relatively free set as just defined is free in the usual sense.

THEOREM 1. *Let G be a connected Lie group of dimension $d \geqslant 1$. Then for every neighborhood U of the identity e in G, there is in U a relatively free set S such that* $\operatorname{card}(S) = \operatorname{card}(\mathbf{R}) \equiv \mathfrak{c}$.

PROOF. Let (B_2, f) provide a chart for a neighborhood contained in U and assume $f(0) = e$; $f^{-1}\colon f(B_2) \to B_2$ gives rise to d coordinate maps $f_i^{-1}\colon f(B_2) \to \mathbf{R}$. We assume $B_2 = \{(x_1, \ldots, x_d)\colon |x_i| < 2\}$ and we set $\overline{B}_1 = \{(x_1, \ldots, x_d)\colon |x_i| \leqslant 1\}$. Finally, let D be the countable Cartesian product of $\overline{B}_1$ with itself. We may assume Lebesgue measure μ_d is normalized in $\mathbf{R}^d$ so that $\mu_d(\overline{B}_1) = 1$; thereby we may introduce product measure μ in D, where, in particular $\mu(D) = 1$. An element of D is a sequence $\{d_\nu\}^\infty$ of d-tuples of real numbers. A sequence $\delta \equiv \{d_\nu\} \in D$ is called *binding* iff, for some word W_{lk} nontrivial for G and at some point in $(\{f(d_\nu)\}_1^\infty)^{\times l} \cap E_l \equiv f(\delta; l)$, $\tilde{W}_{lk} = e$. In the preceding, $\{f(d_\nu)\}_1^\infty$ is the set $f(d_1), f(d_2), \ldots$ in G.

Note that for each $k \in \mathbf{N}$ there are only finitely many words W_{lk} hence finitely many words nontrivial for G. For each word W_{lk} the set of values of $\tilde{W}_{lk}$ on $f(\delta, l)$ is countable. Furthermore, $\tilde{W}_{lk} = e$ iff $f_i^{-1} \circ \tilde{W}_{lk} = 0$, $i = 1, 2, \ldots, d$.

If W_{lk} is nontrivial for G, then $\tilde{W}_{lk} \not\equiv e$ on $G^{\times l}$. Since the map $\tilde{W}_{lk}\colon G^{\times l} \to G$ is analytic and since G and thus also $G^{\times l}$ are connected, Lemma 2 applies and shows that there is no neighborhood on which $\tilde{W}_{lk}$ is constant. In particular, $\tilde{W}_{lk}$ cannot reduce to e throughout $f(B_1)^{\times l}$.

Hence for any word W_{lk} nontrivial for G, the analytic functions

$$f_i^{-1} \circ \tilde{W}_{lk}\underbrace{(f, f, \ldots, f)}_{l\text{-terms}} : B_1^{\times l} \to \mathbf{R}$$

are not all constant. Thus, by Lemma 1,

$$\underbrace{\mu_d \times \mu_d \times \cdots \times \mu_d}_{l\text{-factors}}\{(d_1, d_2, \ldots, d_l)\colon \tilde{W}_{lk}(f(d_1), f(d_2), \ldots, f(d_l)) = e\} = 0.$$

Thus for each set of l distinct indices $\nu_1, \nu_2, \ldots, \nu_l$ and for each word W_{lk} nontrivial for G,

$$\mu\{\delta \equiv \{d_\nu\}\colon \tilde{W}_{lk}(f(d_{\nu_1}), f(d_{\nu_2}), \ldots, f(d_{\nu_l})) = e\} = 0.$$

Hence, for W_{lk} fixed,

$$\mu\{\delta \equiv \{d_\nu\}: \tilde{W}_{lk}(f(d_{\nu_1}), f(d_{\nu_2}), \ldots, f(d_{\nu_l})) = e \text{ for some set of } l \text{ distinct indices } \nu_1, \nu_2, \ldots, \nu_l\} = 0.$$

Hence, if

$$\beta = \{\delta \equiv \{d_\nu\}: \tilde{W}_{lk}(f(d_{\nu_1}), \ldots, f(d_{\nu_l})) = e \text{ for some word } W_{lk} \text{ and some set of } l \text{ distinct indices } \nu_1, \ldots, \nu_l\},$$

then clearly β contains the set of *binding* sequences and $\mu(\beta) = 0$. In particular $D \setminus \beta \neq \varnothing$. Hence, if $\{d_\nu\} \in D \setminus \beta$, then $\{d_\nu\}$ is *not binding*. The set $S_0 \equiv \{f(d_\nu)\}_1^\infty$ leads to the set S promised, as the following lines show.

First note that $\nu_1 \neq \nu_2$, then $f(d_{\nu_1}) \neq f(d_{\nu_2})$. Otherwise for the clearly nontrivial word $W_{22} \equiv u_1 u_2^{-1}$, the substitution $u_1 \to f(d_{\nu_1}), u_2 \to f(d_{\nu_2})$ shows that $\{d_\nu\} \in \beta$, a contradiction. Thus S_0 is infinite.

Second, S_0 is relatively free since if $\tilde{W}_{lk} = e$ at some point of $S_0^{\times l} \cap E_l$ and if W_{lk} is nontrivial for G, then $\{d_\nu\}$ is binding.

Clearly $\mathrm{card}(S_0) = \aleph_0 \equiv \mathrm{card}(\mathbf{N})$. An argument in [3] may be modified as follows to show that there is a relatively free set $S_{\max} \subset U$ and $\mathrm{card}(S_{\max}) = \mathfrak{c}$. In [3] the following lemma is proved: *If* $\mathfrak{F} \equiv \{F_\alpha(x_1, \ldots, x_n)\}$ *is a family of functions analytic on an open cube* $C^0 = \{(x_1, x_2, \ldots, x_n): |x_i - a_i| < \varepsilon_i > 0, i = 1, 2, \ldots, n\}$, *if no* $F_\alpha \equiv 0$ *and if* $\mathrm{card}(\mathfrak{F}) < \mathfrak{c}$, *then there is an n-tuple* $(b_1, b_2, \ldots, b_n)$ *such that, for all* α, $F_\alpha(b_1, \ldots, b_n) \neq 0$. In [3] a transfinite induction establishes the existence of a free subset of cardinality $\mathfrak{c}$ in the rotation group $O(n, \mathbf{R})$ in Euclidean space $\mathbf{R}^n$, $n \geqslant 3$. Used in that development are (a) the existence in $O(3, \mathbf{R})$ of a free set of two elements (this fact was proved by R. Robinson [6] in the context of F. Hausdorff's result [5] on rotations in $\mathbf{R}^3$: $O(3, \mathbf{R}) \supset F/N$, F the free group on two generators a, b, and N the normal subgroup generated by a^2 and b^3; Robinson showed that in F the free group generated by $abab$ and ab^2ab^2 meets N in the identity; thus $O(3, \mathbf{R})$ contains a free group on two generators); (b) Kurosh's theorem showing that a free group contains an *infinite* free set. In both [3] and [5], as in the present discussion, use is made of the theory of analytic functions of several real variables.

The following lines are in essence a Zorn's-lemma version of the final argument in [3]. In the nonempty set IRF of *infinite relatively free* subsets of U there is, via Zorn's lemma applied to IRF partially ordered by inclusion, a maximal set S. We show $\gamma \equiv \mathrm{card}(S) = \mathfrak{c}$. Clearly $\gamma \leqslant \mathfrak{c}$. If $\gamma < \mathfrak{c}$, for each word W_{lk} nontrivial for G, each substitution, whereby $l - 1$ symbols are replaced by fixed and distinct elements $f(d_1), \ldots, f(d_{l-1})$ of S and the other symbol is replaced by an arbitrary $y \in f(B_2)$, yields a function $\tilde{W}_{lk}^y$ that depends analytically on y, i.e., the functions

$$k_i(y) \equiv f_i^{-1} \circ \tilde{W}_{lk}^y \circ (f, f, \ldots, f)(d_1, d_2, \ldots, d_{l-1}, y)$$

are analytic maps of B_2 into $\mathbf{R}$. For at least one i, $k_i(y)$ is not identically 0 in B_2 since if y is in S, but $y \not\in (f(d_1), f(d_2), \ldots, f(d_{l-1}))$, then $\tilde{W}_{lk}^y \neq e$. Here the fact that S is infinite is used in an essential way.

The cardinality of the set of all such functions (derived from the (countable) set of all words nontrivial for G, the set of all one-to-one substitutions of the kind described (here the cardinality is not more than $\gamma^l = \gamma$) and all $i = 1, 2, \ldots, d$ as needed) has cardinality $< \mathfrak{c}$. Thus by the lemma of [3] as quoted above, this set of functions has a common nonzero: $b_1^0, b_2^0, \ldots, b_n^0$. If $y_0 = f(b_1^0, \ldots, b_n^0)$ and if g is any element of the group generated by S, then $y_0 \neq g$ since otherwise $y_0^{-1}g$ is simultaneously some $\tilde{W}_{lk}^{y_0}$ and also e, a contradiction. Thus, in particular $y_0 \notin S$ and $S \cup \{y_0\} \supsetneqq S$ is a relatively free set since every $\tilde{W}_{lk}^{y_0} \neq e$. The maximality of S is contradicted, whence $\gamma = \mathfrak{c}$, and S is the set promised in the theorem.

It should be noted, in view of Paul Cohen's results and the subsequent developments of several others working in set theory, that the de Groot-Dekker lemma is a trivial consequence of Lemma 1 of the present paper if one accepts the continuum hypothesis as an axiom.

Indeed, the family $\mathfrak{F}$ is countable since card$(\mathfrak{F}) < \mathfrak{c}$ and the continuum hypothesis is an axiom. Thus the union N of the sets N_α of zeros of the F_α is, by Lemma 1, the *countable* union of null sets, hence N is itself a null set. Hence $C^0 \setminus N$ is nonempty and any $(b_1, b_2, \ldots, b_n)$ in $C^0 \setminus N$ satisfies: $F_\alpha(b_1, b_2, \ldots, b_n) \neq 0$, all α.

Variations on this theme when the continuum hypothesis is *denied* are also relevant; e.g., there are set-theory models in which the continuum hypothesis is false but in which Lebesgue measure is "$\mathfrak{c}$-additive", i.e., if $\{N_\alpha\}$ is a family of null sets and card$\{\alpha\} < \mathfrak{c}$ then $\bigcup_\alpha N_\alpha$ is a null set. Thus, in the notation above, $N = \bigcup_\alpha N_\alpha$ is a null set even if $\{\alpha\}$ is not countable, but card$\{\alpha\} < \mathfrak{c}$. The argument just given then shows $F_\alpha \neq 0$ on $C^0 \setminus N$.

In another direction there is a characterization of the words that may serve as identities for a Lie group G, $\dim G \geqslant 1$. If $\tilde{W}_{lk} \equiv e$ on $G^{\times l}$, then $\sum_{w_i = u_j} \varepsilon_i = 0, j = 1, 2, \ldots, l$. Indeed, otherwise, for some $p \in \mathbf{N}$, $g^p \equiv e$ for all g in G. Hence for any one parameter subgroup $\{g(t)\colon -\infty < t < \infty\} \equiv H_g$, $g(pt) \equiv e$ for all t. Thus any tangent vector, at e to H_g, is 0, in contradiction of a basic property of Lie groups.

On the other hand, if $\sum_{w_i = u_j} \varepsilon_i = 0$, $j = 1, 2, \ldots, l$, then every Abelian group A, whether it is a Lie group or not, is one for which $\tilde{W}_{lk} \equiv e$ on $A^{\times l}$. We conclude: *a word W_{lk} is admissible as a universal restriction on a Lie group G, i.e., $\tilde{W}_{lk} \equiv e$ on $G^{\times l}$, iff $\sum_{w_i = u_j} \varepsilon_i = 0, j = 1, 2, \ldots, l$.*

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