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SINGULAR PERTURBATIONS AND THE TRANSITION FROM THIN PLATE TO MEMBRANE

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ABSTRACT. The equation

$$\frac{Eh^2}{12(1-\sigma^2)}\Delta^2 w - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(\sigma_{ij}\frac{\partial w}{\partial x_j}\right) = f$$

describing the normal displacement w of a thin elastic plate of thickness h under uniform tension in equilibrium is considered. It is shown that if the displacement and its normal derivative on the edge of the plate are bounded uniformly with respect to h then the solution u of the membrane equation

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(\sigma_{ij}\frac{\partial u}{\partial x_j}\right) = f$$

with the same boundary values as w approximates the displacement throughout the plate in the L^2 sense. Herein, the rate

$$(*) \qquad \iint_\Omega |w(x) - u(x)|^2\,dx \in Ch^2 \iint_\Omega |f(x)|^2\,dx$$

is given, where C is a constant independent of h and f, and Ω in the face of the plate. This extends the results of A. Friedman [6] and F. John [10] up to the boundary and improves the rate of convergence in $(*)$ given by J. L. Lions [12] and W. M. Greenlee [7] from h to h^2.

The equation describing the normal displacement $w(x_1, x_2)$ of a thin elastic plate of thickness h in equilibrium is [11]

$$\text{(I)} \qquad \frac{Eh^2}{12(1-\sigma^2)}\Delta^2 w - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(\sigma_{ij}\frac{\partial w}{\partial x_j}\right) = f.$$

Here as usual, E denotes Young's modulus, σ Poisson's ratio, σ_{ij} are the components of the stress in the plate and f is the external force per unit area. Since the effect of the forces or constraints acting on the edge of the plate can be achieved by replacing w by $w - \Phi$ we may assume that the plate does not undergo any deformation at the edge; thus the displacement is caused solely by forces acting on the faces of the plate. The corresponding displacement of a membrane is governed by the second order equation [11]

$$\text{(II)} \qquad -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(\sigma_{ij}\frac{\partial u}{\partial x_j}\right) = f.$$

As mentioned in [10], replacing (I) by (II) can in general only be justified

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when the plate is under tension, i.e. when the σ_{ij} form the coefficients of a positive definite quadratic form. The special case $f = 0$ and w and its derivative is given on the edge of the plate was considered by F. John [10]. For general constant tension σ_{ij} he obtains the interior estimate

$$w(\mathbf{y}) - u(\mathbf{y}) = O\left(\frac{h^2}{r^2} W\right) \qquad \left(W = \sup_{\mathbf{x}\in\Omega} |w(\mathbf{x})|\right) \tag{0.1}$$

where $u(y)$ is a solution of (II) determined by the boundary values of w and its derivatives up to order 3 on the edge of the plate. Here r denotes the distance from $\mathbf{y}$ to the edge. Thus u is also a function of h. In this paper we show that if the displacement and its normal derivatives on the edge of the plate are bounded uniformly with respect to h then the membrane solution with the same boundary values as w approximates the displacement throughout the plate in the L^2 sense. Herein, the rate

$$\iint_\Omega |w(\mathbf{x}) - u(\mathbf{x})|^2 \, d\mathbf{x} \leqslant Ch^2 \iint_\Omega |f(\mathbf{x})|^2 \, d\mathbf{x} \tag{0.2}$$

is given, where C is a constant independent of h and f, and Ω is the face of the plate.

Similar estimates were derived by J. L. Lions [12] where the $H^1(\Omega)$ norm of $w - u$ is shown to be $O(h^{1/2}\|f\|_{L^2(\Omega)})$. A. Friedman [6] derived (0.2) in the *interior* of Ω; W. Greenlee [7] obtained results similar to [12], C. Bardos and D. and H. Brezis [2] prove the strong convergence in $L^2(\Omega)$ of w to u; they do not obtain, however, estimates similar to (0.2). M. I. Vishik and L. A. Liusternik [14] expand w asymptotically in h under some differentiability assumptions on f, but they do not estimate the rate of convergence of w to u.

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1. **Preliminary results and estimates in a half plane**. Let Ω be a bounded domain in R^2 with boundary $\partial\Omega$ in the class C^∞. We denote by n the outward normal to $\partial\Omega$. We may assume that $\sigma_{11} = 1$ and use $\varepsilon = Eh^2/12(1 - \sigma^2)$. Set

$$L = \varepsilon\Delta^2 - \sum_{i,j=1}^{2} \sigma_{ij} \frac{\partial^2}{\partial x_i \partial x_j}, \tag{1.1}$$

$$M = -\sum_{i,j=1}^{2} \sigma_{ij} \frac{\partial^2}{\partial x_i \partial x_j}. \tag{1.1'}$$

Let $w(\mathbf{x}) = w(x_1, x_2)$ be the solution of

$$Lw = f \quad \text{in } \Omega, \tag{1.2}$$

$$w = \partial w/\partial n = 0 \quad \text{on } \partial\Omega \tag{1.2'}$$

and let $u(\mathbf{x})$ be the solution of

$$Mu = f \quad \text{in } \Omega, \tag{1.3}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.3'}$$

Let $\phi(\mathbf{x})$ be a smooth function in Ω which is equal to the distance from the boundary near $\partial\Omega$ and $\phi = 1$ away from $\partial\Omega$. We use $H^k(\Omega)$ to denote the

Sobolev space [5] of all square integrable functions in Ω that have square integrable partial derivatives up to order k in Ω, and for $v \in H^k(\Omega)$ the norm is given by

$$\|v\|^2_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \iint_\Omega |D^\alpha v|^2 \, d\mathbf{x}.$$

LEMMA 1.1. *There exists a positive constant C such that*

$$\varepsilon\|w\|^2_{H^2(\Omega)} + \|w\|^2_{H^1(\Omega)} \leq C\|f\|^2_{L^2(\Omega)}, \tag{1.4}$$

$$\varepsilon^2\|\phi w\|^2_{H^4(\Omega)} + \|\phi u\|^2_{H^2(\Omega)} \leq C\|f\|^2_{L^2(\Omega)} \tag{1.5}$$

for all $\varepsilon > 0$.

PROOF. Inequality (1.4) is easily obtained using integration by parts (cf. D. Huet [9] or A. Friedman [6]). To obtain (1.5) we multiply equation (1.1) by $\phi^2 Mw$ and integrate by parts. Using (1.4) we obtain

$$\|\phi Mw\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

It follows from [13], [1], or [3] and (1.4) that

$$\|\phi w\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

Now we multiply (1.1) by ϕ and obtain

$$\varepsilon\|\phi\Delta^2 w\|_{L^2(\Omega)} \leq C(\|\phi Mw\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}).$$

Using [13], [1], [3] again we get (1.5).

COROLLARY. *Under the assumptions of Lemma* 1.1

$$\varepsilon\|w\|_{H^3(\Omega)} \leq C\|f\|_{L^2(\Omega)} \tag{1.6}$$

and

$$\varepsilon^{1/2}\|\phi w\|_{H^3(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \tag{1.7}$$

PROOF. Using (1.4), (1.1) and the uniform ellipticity of Δ^2 we see [5] that

$$\|w\|_{H^2(\Omega)} \leq C\varepsilon^{-1/2}\|f\|_{L^2(\Omega)} \quad \text{and} \quad \|w\|_{H^4(\Omega)} \leq C\varepsilon^{-3/2}\|f\|_{L^2(\Omega)}.$$

Hence using interpolation [12] we get (1.6). Similar arguments lead to (1.7).

Let R^2_+ be the half plane $\{(x_1, x_2) \in R^2 | x_1 \in R^1, x_2 > 0\}$.

LEMMA 1.2. *Assume $f(x) \in L^2(R^2_+)$ and $f(x_1, x_2) = 0$ for $x_2 > K$. Let w be the solution of* (1.1) *in R^2_+ satisfying the Dirichlet boundary conditions*

$$w(x_1, 0) = \partial w(x_1, 0)/\partial x_2 = 0, \quad w \text{ is bounded in } R^2_+, \tag{1.8}$$

and let $u(x)$ be the solution of (1.2) *in R^2_+ satisfying*

$$u(x_1, 0) = 0, \quad u \text{ is bounded in } R^2_+. \tag{1.9}$$

Then there exist positive constants ε_0 and C such that

$$\|w - u\|_{L^2(R^2_+)} \leq C\varepsilon^{1/2}\|f\|_{L^2(R^2_+)} \tag{1.10}$$

for all $0 < \varepsilon \leq \varepsilon_0$, C is independent of ε and f but depends on K.

PROOF. Assume $f \in C_0^\infty(R_+^2)$. Using the Fourier transform in x_1 we obtain for w and h the ordinary differential equations

$$(1.11)\quad \left[\varepsilon\left(-\xi_1^2 + \frac{\partial^2}{\partial x_2^2}\right)^2 - \left(-\xi_1^2 + 2\sigma_{12} i\xi_1 \frac{\partial}{\partial x_2} + \sigma_{22}\frac{\partial^2}{\partial x_2^2}\right)\right]\hat{u}(\xi_1, x_2) = \hat{f}(\xi_1, x_2)$$

and

$$(1.12)\quad \left[-\xi_1^2 + 2\sigma_{12} i\xi_1 \frac{\partial}{\partial x_2} + \sigma_{22}\frac{\partial^2}{\partial x_2}\right]\hat{u}(\xi_1, x_2) = \hat{f}(\xi_1, x_2).$$

The characteristic equation for (1.11) is

$$(1.13)\quad \varepsilon(\rho^2 - \xi_1^2)^2 - (\sigma_{22}\rho^2 + 2\sigma_{12} i\xi_1\rho - \xi_1^2) = 0.$$

Setting $\rho = \xi_1 r$ we obtain

$$(1.14)\quad \delta(r^2 - 1)^2 - (\sigma_{22} r^2 + 2\sigma_{12} ir - 1) = 0$$

where $\delta = \varepsilon\xi_1^2$.

We consider the three cases: (i) $\delta < \delta_1$, (ii) $\delta > \delta_2$ and (iii) $\delta_1 \leqslant \delta \leqslant \delta_2$, where δ_1 is a sufficiently small positive number and δ_2 is a sufficiently large number independent of ε and ξ_1. In case (i) we set $r = s^{-1}$ so that (1.13) becomes

$$\delta(1 - s^2)^2 - (\sigma_{22} s^2 + 2\sigma_{12} is^3 - s^4) = 0.$$

Since the equation $\sigma_{22} s^2 + 2bis^3 - s^4 = 0$ has the double root $s = 0$ and two simple roots s_1 and s_2, we have [8] the expansions

$$s = \pm\delta^{1/2}[1 + o(1)] \quad \text{as } \delta \to 0$$

and

$$s = s_j[1 + O(\delta)] \quad \text{as } \delta \to 0, j = 1, 2.$$

Therefore there exists a positive constant δ_1 such that the roots ρ_j of (1.11) have the representation

$$\rho_1 = \xi_1\left(i\sigma_{12} - (\sigma_{22} - \sigma_{12}^2)^{1/2} + \delta\alpha\right) \equiv \xi_1(ib - D + \alpha\delta) \equiv ib_\varepsilon\xi_1 - D_\varepsilon\xi_1,$$

$$\rho_2 = \xi_1(ib + D + \delta\alpha) \equiv ib_\varepsilon\xi_1 + D_\varepsilon\xi_1,$$

$$\rho_3 = \varepsilon^{-1/2}(1 + \delta\beta)\operatorname{sgn}\xi_1 \equiv ib'_\varepsilon\xi_1 - D'_\varepsilon\xi_1,$$

$$\rho_4 = -\varepsilon^{-1/2}(1 + \delta\beta)\operatorname{sgn}\xi_1 \equiv ib'_\varepsilon\xi_1 + D'_\varepsilon\xi_1,$$

where α and β are bounded functions of ε and ξ_1. In case (ii) we get similarly

$$\rho_j = \pm\varepsilon^{-1/2}(1 \pm \delta^{-1/2}\gamma), \qquad j = 1, 2, 3, 4,$$

where γ is a bounded function of ε and ξ_1.

Finally in case (iii)

$$\rho_j = \xi_1 r_j, \qquad j = 1, 2, 3, 4,$$

where r_j are bounded functions of ε and ξ_1. We have $\rho_1 = -\bar{\rho}_2$ and $\rho_3 = -\bar{\rho}_4$. The solution u of (1.2), (1.9) is given by

$$\hat{u}(\xi_1, x_2) = \int_0^\infty G(\xi_1, x_2, t)\hat{f}(\xi_1, t)\, dt$$

where

$$G(\xi_1, x_2, t) = -[\exp(ib|\xi_1|\,|x_2 - t| - D|\xi_1|x_2)](\sinh D|\xi_1|t/D|\xi_1|)$$

if $x_2 \geqslant t$, and for $x_2 < t$, G is defined by the relation

$$G(\xi_1, x_2, t) = \overline{G(\xi_1, t, x_2)}\,.$$

The solution w of (1.1), (1.8) is given by

$$\hat{w}(\xi_1, x_2) = \int_0^\infty G_\varepsilon(\xi_1, x_2, t)\hat{f}(\xi_1, t)\, dt + \int_0^\infty H_\varepsilon(\xi_1, x_2, t)\hat{f}(\xi_1, t)\, dt$$

where the functions G_ε and H_ε are defined by

$$\begin{aligned} G_\varepsilon(\xi_1, x_2, t) &= -[\exp(ib_\varepsilon|\xi_1|\,|x_2 - t| - D_\varepsilon|\xi_1|x_2)]/|\rho_3 - \rho_1|^2\varepsilon \\ &\quad \cdot\left[\frac{(\rho_4 - ib_\varepsilon|\xi_1|)}{(\rho_4 - \rho_1)}\,\frac{\sinh D_\varepsilon|\xi_1|t}{D_\varepsilon|\xi_1|} - \frac{\cosh D_\varepsilon|\xi_1|t}{\rho_4 - \rho_1}\right] \quad \text{if } x_2 \geqslant t, \end{aligned}$$

$$\begin{aligned} H_\varepsilon(\xi_1, x_2, t) &= G'_\varepsilon(\xi_1, x_2, t) + \frac{\exp(-|\xi_1|(D_\varepsilon + D'_\varepsilon)t)}{|\rho_1 - \rho_3|^2|\rho_1 - \rho_4|^2} \\ &\quad \cdot[(\rho_4 - \rho_1)\exp(i|\xi_1|(b'_\varepsilon x_2 - b_\varepsilon t)) \\ &\quad - (\bar{\rho}_4 - \bar{\rho}_1)\exp(-i|\xi_1|(b'_\varepsilon t - b_\varepsilon x_2))] \quad \text{if } x_2 \geqslant t, \end{aligned} \tag{1.15}$$

and for $x_2 < t$ they are defined by the relations

$$G_\varepsilon(\xi_1, x_2, t) = \overline{G_\varepsilon(\xi_1, t, x_2)} \quad \text{and} \quad H_\varepsilon(\xi_1, x_2, t) = \overline{H_\varepsilon(\xi_1, t, x_2)}\,.$$

The function G'_ε is given by (1.15) with the roles of ρ_1 and ρ_3 exchanged. We consider case (i) first. A typical summand of $\hat{w} - \hat{u}$ is

$$\begin{aligned} l_1 &= \int_0^{x_2}\left[\frac{\exp(ib_\varepsilon|\xi_1|\,|x_2 - t| - D_\varepsilon|\xi_1|x_2)}{(\rho_4 - \rho_1)|\rho_3 - \rho_1|^2\varepsilon}\cosh D_\varepsilon|\xi_1|t\right]\hat{f}(\xi_1, t)\, dt \\ &\equiv \int_0^{x_2} W(\xi_1, x_2, t)\hat{f}(\xi_1, t)\, dt \end{aligned}$$

The denominator is equal to $\varepsilon^{-1/2}[1 + O(\delta)]$ if δ is small, $\delta \leqslant \delta_1$ say, therefore

$$|l_1| \leqslant C\varepsilon^{1/2}\int_0^{x_2}[\exp(-D_\varepsilon|\xi_1|x_2)][\cosh(D_\varepsilon|\xi_1|t)]|\hat{f}(\xi_1, t)|\, dt,$$

and similarly,

$$\begin{aligned} |l_2| &= \left|\int_{x_2}^\infty \overline{W(\xi_1, t, x_2)}\,\hat{f}(\xi_1, t)\, dt\right| \\ &\leqslant C\varepsilon^{1/2}\int_{x_2}^\infty[\exp(-D_\varepsilon|\xi_1|t)]\,[\cosh D_\varepsilon|\xi_1|x_2]|\hat{f}(\xi_1, t)|\, dt. \end{aligned}$$

Thus

$$|l_1 + l_2| \leqslant C\varepsilon^{1/2} \int_0^K K(x_2, t) |\hat{f}(\xi_1, t)| \, dt$$

since $\operatorname{supp} f(x_1, x_2) \subset \{0 \leqslant x_2 \leqslant K\}$. To estimate $\|l_1 + l_2\|_{L^2(R_+^2)}$ we rely on the following principle: *if* $\int |K(x, y)| \, dx \leqslant C_1$ *and* $\int |K(x, y)| \, dy \leqslant C_2$ *then*

$$\int \left| \int K(x, y) f(y) dy \right|^2 dx \leqslant C_1 C_2 \int |f(x)|^2 \, dx,$$

which follows easily from the Riesz-Thorin convexity theorem [4]. Applying Plancherel's formula and the above principle we get

$$\int_{-\infty}^{\infty} \left(\int_0^{\infty} |(l_1 + l_2)(\xi_1, x_2)| \, dx_2 \right)^2 d\xi_1 \leqslant C_1 C_2 \int_{-\infty}^{\infty} \int_0^{\infty} |f(x_1, x_2)|^2 \, dx_2 \, dx_1$$

where

$$C_1 = \sup_{\xi_1, x_2} \int_0^K |K(x_2, t)| \, dt$$

and C_2 is defined similarly. Direct computations show that

$$C_1, C_2 \leqslant C\varepsilon^{1/2} \sup_{\xi_1} \frac{1 - e^{-K|\xi_1|}}{|\xi_1|} = C\varepsilon^{1/2}.$$

The other terms of $\hat{w} - \hat{u}$ are similarly estimated by $C\varepsilon^{1/2}$. In cases (ii) and (iii) all terms are bounded by $C\varepsilon^{1/2}$ in $L^2(R_+^2)$ and C is independent of K. In cases (iii) we use the fact $|\xi_1| \geqslant \delta_1 \varepsilon^{-1/2}$ and $|r_j - r_k| \geqslant c > 0$ if $j \neq k$. This concludes the proof of Lemma 1.2.

2. Estimates near the boundary and main results.

LEMMA 2. *Let w and u be the solutions of* (1.1) *and* (1.3) *in Ω satisfying the homogeneous Dirichlet boundary conditions. Then there exists a neighborhood U of $\partial\Omega$ in $\overline{\Omega}$ such that*

$$\|w - u\|_{L^2(U)} \leqslant C\varepsilon^{1/2} \|f\|_{L^2(\Omega)} + C\|w - u\|_{L^2(\Omega - U)}, \tag{2.1}$$

where C and U are independent of ε and f.

PROOF. For any point $\mathbf{x}_0 \in \partial\Omega$ we can find a neighborhood U_0 of $\mathbf{x}_0$ in $\overline{\Omega}$ and a smooth mapping of U_0 into $B_r = \{\mathbf{x} \in R^2 | \, |x_1| < r, \, 0 \leqslant x_2 < r^2\}$ so that $\overline{U}_0 \cap \partial\Omega$ is mapped onto $B_r \cap \{x_2 = 0\}$ and such that $\mathbf{x}_0$ is mapped into the origin. Then using an affine transformation of the coordinates we can write L and M in B_r as follows.

$$\begin{aligned} \tilde{L} = \varepsilon\Delta^2 - \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \varepsilon[x_1 N(\mathbf{x}, D) + x_2 P(\mathbf{x}, D) + Q(\mathbf{x}, D)] \\ + x_1 R(\mathbf{x}, D) + x_2 S(\mathbf{x}, D) + T(\mathbf{x}, D), \end{aligned} \tag{2.2}$$

$$\tilde{M} = - \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + x_1 R(\mathbf{x}, D) + x_2 S(\mathbf{x}, D) + T(\mathbf{x}, D) \tag{2.3}$$

where the quadratic form corresponding to a_{ij} is positive definite, a_{ij} is constant and N, P, Q, R, S, T are partial differential operators with coefficients in $C_0^\infty(B_r)$ of orders 4, 4, 3, 2, 2 and 1 respectively. Let ψ be a

function in $C_0^\infty(B_r)$ such that $\psi = 1$ in $B_{r/2}$. Then the functions ψw and ψu satisfy equations of the form

$$L\psi w = f_{1,\varepsilon}, \tag{2.4}$$

$$M\psi u = f_1 \tag{2.5}$$

where L and M are defined by (1.1) and (1.1′) respectively (σ_{ij} is replaced by a_{ij}) and $f_{1,\varepsilon}$ and f_1 are defined by (2.2)–(2.5). We have to estimate terms of the form $\varepsilon x_1 Nw$, $\varepsilon x_2 Pw$, etc. We may assume that $Nw = \sum_{|\alpha|\leqslant 4} D^\alpha(a^\alpha(\mathbf{x})w)$ so that

$$\int_0^\infty G_\varepsilon(\xi_1, x_2, t)(x_1 Nw)\hat{}(\xi_1, t)\,dt = -i\int_0^\infty G_\varepsilon(\xi_1, x_2, t)\frac{\partial}{\partial \xi_1}(Nw)\hat{}(\xi_1, t)\,dt$$

$$= i\int_0^\infty \frac{\partial}{\partial \xi_1} G_\varepsilon(\xi_1, x_2, t)\sum_{|\alpha|\leqslant 4}\left(i\xi_1\frac{\partial}{\partial x_2}\right)^\alpha (a^\alpha w)\hat{}(\xi_1, t)\,dt.$$

Since $\xi_1^2\partial(G_\varepsilon + H_\varepsilon)/\partial\xi_1$ is a bounded operator and Lemma 1.1 shows that $\varepsilon^{1/2}\|w\|_{H^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}$, we see that all summands of $\varepsilon x_1 Nw$ containing two or more derivatives with respect to x_1 contribute to $L^{-1}(\varepsilon x_1 Nw)$ a quantity that is bounded by $C\varepsilon^{1/2}\|f\|_{L^2(\Omega)}$. The functions $G_\varepsilon + H_\varepsilon$, $\partial(G_\varepsilon + H_\varepsilon)/\partial t$ and their ξ_1 derivative vanish for $x_2 = 0$, therefore we can integrate by parts expressions containing three or four derivatives of w with respect to t. Since $\partial^3(G_\varepsilon + H_\varepsilon)/\partial\xi_1\partial t^2$ is a bounded operator in $L^2(B_r)$ and $\psi w \in C_0^\infty(B_r)$, we see that the contribution of such derivatives to $L^{-1}(\varepsilon x_1 Nw)$ is bounded by $C\varepsilon^{1/2}\|f\|_{L^2(\Omega)}$. To estimate $\varepsilon x_2 Pw$ we use similar arguments and (1.7), and for εQw we use integration by parts and (1.6). Writing $\psi w = L^{-1}f_{1,\varepsilon}$ and $\psi u = M^{-1}f_1$ we see that we have to estimate expressions of the form

$$(L^{-1} - M^{-1})x_1 R(\mathbf{x}, D)w \quad \text{and} \quad M^{-1}x_1 R(\mathbf{x}, D)(w - u),$$

etc. The first expression can be estimated by $C\varepsilon^{1/2}\|f\|_{L^2(\Omega)}$ using Lemma 1.2 and (1.5). To estimate the second expression we note that the function ψ can be chosen so that it is independent of x_2 in $0 \leqslant x_2 \leqslant r^2/4$, and $|\partial\psi/\partial x_1| \leqslant C/r$, $|\partial^2\psi/\partial x_1^2| \leqslant C/r^2$. Using integration by parts and the fact that G vanishes on the boundary $x_2 = 0$ we obtain the estimate

$$\|w - u\|_{L^2(U_0')} \leqslant C\varepsilon^{1/2}\|f\|_{L^2(\Omega)} + Cr\|w - u\|_{L^2(U_0)}$$

where C is independent of ε and r. The neighborhood U_0' of x_0 is the inverse image of $B_{r/2}$. The boundary $\partial\Omega$ can be covered by a finite number of neighborhoods $\{U_j'\}$ so that the number of intersecting neighborhoods $\{U_j\}$ at each point is less than 4 (cf. [15]). Setting $U = \bigcup_j U_j'$ and using a partition of unity we obtain

$$\|w - u\|_{L^2(U)} \leqslant C_1 r\|w - u\|_{L^2(U)} + C_2\|w - u\|_{L^2(\Omega - U)}$$

where C_1 is independent of r; hence (2.1) follows.

The main result of this paper is

THEOREM 2.1. *Let Ω be a bounded domain in R^2 with smooth boundary $\partial\Omega$ and let w and u be the solutions of* (1.2), (1.2′) *and* (1.3), (1.3′) *respectively. Then there exists a constant C such that for all $\varepsilon > 0$ and $f \in L^2(\Omega)$,*

$$\|w - u\|_{L^2(\Omega)} \leqslant C\varepsilon^{1/2}\|f\|_{L^2(\Omega)}. \tag{2.6}$$

PROOF. Using the interior estimates of A. Friedman [6], we have for any subdomain Ω_0 of Ω, such that $\overline{\Omega}_0 \subset \Omega$, the estimate

$$\|w - u\|_{L^2(\Omega_0)} \leqslant C\varepsilon^{1/2}\|f\|_{L^2(\Omega)}, \tag{2.7}$$

where C is a constant independent of ε and f, but C depends on $\operatorname{dist}(\Omega_0, \partial\Omega)$. Combining (2.1) and (2.7), inequality (2.6) follows.

The following results can be proved by the same methods and the results of A. Friedman [6].

THEOREM 2.2. *Let w and u be the solutions of* (1.2) *and* (1.3) *with boundary conditions*

$$w = \partial^2 w/\partial n^2 = 0 \quad \text{on } \partial\Omega, \tag{2.8}$$

or

$$w = \Delta w = 0 \quad \text{on } \partial\Omega, \tag{2.9}$$

or

$$w = \partial\Delta w/\partial n = 0 \quad \text{on } \partial\Omega \tag{2.10}$$

and

$$u = 0 \quad \text{on } \partial\Omega. \tag{2.11}$$

Then (2.6) *holds.*

THEOREM 2.3. *Let Q_T be the cylinder $Q_T = \Omega \times (0, T]$, $B_T = \partial\Omega \times (0, T]$. Let w be the solution of $\partial w/\partial t + Lw = f(x, t)$ in Q_T satisfying* (1.2′), (2.8), (2.9) *or* (2.10) *and* (2.11) *on B_T, $w(x, 0) = 0$ in Ω, and let u be the solution of $\partial u/\partial t + Mu = f(\mathbf{x}, t)$ in Q_T satisfying* (2.11) *on B_T and $u(\mathbf{x}, 0) = 0$ in Ω. Then*

$$\left(\int_0^T\int_\Omega |w(\mathbf{x}, t) - u(\mathbf{x}, t)|^2\, d\mathbf{x}\, dt\right)^{1/2} \leqslant C\varepsilon^{1/2}\left(\int_0^T\int_\Omega |f(\mathbf{x}, t)|^2\, d\mathbf{x}\, dt\right)^{1/2}.$$

This result improves the results of J. L. Lions [12, p. 287], where

$$\|w - u\|_{L^2(Q_T)} = O(\varepsilon^{1/4}),$$

and A. Friedman [6] by extending his estimates up to the boundary.

REMARK. The estimate (2.6) is sharp and can be verified in the case of a clamped, uniform circular plate. It is probably not sharp in case boundary conditions (2.8), (2.9) or (2.10) replace (1.2′). The method of the present paper can be used to obtain sharper results for the various boundary value problems.

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