A GENERALIZATION OF THE HAHN-MAZURKIEWICZ THEOREM

L. E. WARD, JR.

Abstract. It is proved that if a Hausdorff continuum $X$ can be approximated by finite trees (see the text for definition) then there exists a (generalized) arc $L$ and a continuous surjection $\varphi: L \to X$.

1. Introduction. The celebrated Hahn-Mazurkiewicz theorem, first proved about 1914 [4], [8], asserts that a Peano continuum is the image of $[0, 1]$ under some continuous mapping. Subsequent attempts to generalize the theorem to the nonmetric setting proved unavailing, and in 1960 Mardešić [6] described a locally connected Hausdorff continuum which is not arcwise connected (in the generalized sense) and hence is not the continuous image of any arc. Later Cornette and Lehman [3] exhibited a simpler example with the same properties. The possibility remained that an arcwise connected, locally connected continuum is the continuous image of some arc, but in [7] Mardešić and Papić showed that any product of continua which is the continuous image of an arc is necessarily metrizable. Consequently, even such a nice continuum as $L \times [0, 1]$, where $L$ is the "long arc", is not the continuous image of an arc. Later results of Treybig [12], [13], A. J. Ward [15] and Young [19] elaborated on this theme.

Quite recently some affirmative results have appeared. Cornette [2] proved that a tree is the continuous image of some arc, and the author [17] has extended this to rim-finite continua. Different proofs of these results have been found independently by Pearson [10], [11].

In this paper we prove a generalization of the Hahn-Mazurkiewicz theorem which includes all of the aforementioned affirmative results.

We recall some terminology. A continuum is a compact, connected Hausdorff space. An arc is a continuum with exactly two noncutpoints. A tree is a continuum in which each pair of distinct points can be separated by some point. A finite tree is a tree with only finitely many endpoints.

A continuum $X$ can be approximated by finite trees if there exists a family $\mathcal{T}$ of finite trees such that

1. $\mathcal{T}$ is directed by inclusion,
2. $\bigcup \mathcal{T}$ is dense in $X$,
3. if $\mathcal{U}$ is an open cover of $X$ then there exists $T(\mathcal{U}) \in \mathcal{T}$ such that if
Our principal result is the following.

**Theorem 1.** If $X$ is a continuum which can be approximated by finite trees then there exists an arc $L$ and a continuous surjection $\varphi: L \to X$.

2. **Proof of Theorem 1.**

**Lemma 1.** If $\{T_\alpha, r_\beta\}$ is an inverse system of trees and if the bonding mappings $r_\beta$ are monotone, then $T_\infty = \text{inv lim} \{T_\alpha, r_\beta\}$ is a tree.

**Proof.** Nadler [9, Theorem 3] has shown that $T_\infty$ is hereditarily unicoherent, and Capel [1] proved that $T_\infty$ is locally connected. Hence [16, Theorem 9], $T_\infty$ is a tree.

**Lemma 2.** If $T_1$ and $T_2$ are trees with $T_1 \subset T_2$, then there exists a retraction $r: T_2 \to T_1$ which is monotone. Moreover, if $C$ is a component of $T_2 - T_1$ then $C$ has one-point boundary $x(C)$ and $r(C) = x(C)$.

**Proof.** If $C$ is a component of $T_2 - T_1$ then, by the hereditary unicoherence of trees, $\overline{C} \cap T_1$ is connected. Suppose $\overline{C} \cap T_1$ contains distinct elements $x$ and $y$; then there are connected neighborhoods $U_x$ and $U_y$ of $x$ and $y$, respectively, such that $U_x$ and $U_y$ are disjoint. Since $C$ is an open set, we can invoke a standard chaining argument to show the existence of a continuum $K$ which is contained in $C$ and which meets both $U_x$ and $U_y$. If we define $P = U_x \cup K \cup U_y$ and $Q = \overline{C} \cap T_1$, then $P$ and $Q$ are subcontinua of $T_2$, $P \cap Q \subset (U_x \cup U_y)$, and $P \cap Q$ meets both $U_x$ and $U_y$. This contradicts the hereditary unicoherence of the tree $T_2$, and hence $\overline{C} \cap T_1 = \overline{C} - C$ consists of a single point, $x(C)$. Define $r: T_2 \to T_1$ by $r|T_1 = 1$ and $r(C) = x(C)$ for each component $C$ of $T_2 - T_1$. It is straightforward to verify that $r$ is continuous. Finally, $r$ is monotone because, for each $x \in T_2$,

$$r^{-1}(x) = \{x\} \cup \bigcup \{ C: C \text{ is a component of } T_2 - T_1 \text{ and } \overline{C} \cap T_1 = \{x\} \} ,$$

which is a connected set.

For the remainder of this section let $X$ be a continuum which is approximated by the family $\mathcal{T}$ of finite trees. Then the system $\mathcal{T} = \{T_\alpha, r_\beta\}$ is an inverse system with monotone bonding maps, and hence $T_\infty = \text{inv lim} \mathcal{T}$ is a tree.

**Lemma 3.** If $(x_\alpha) \in T_\infty$ then $(x_\alpha)$ is a convergent net in $X$.

**Proof.** Let $p$ be a cluster point of the net $(x_\alpha)$ and suppose $V$ is an open set containing $p$. There exists a finite open cover $\beta$ of $X$ such that if $p \in U \in \beta$ then $\text{Star}(U, \beta) \subset V$. By hypothesis there exists $T_\beta \in \mathcal{T}$ such that if $T_\beta \subset T_\alpha \in \mathcal{T}$ and if $C$ is a component of $T_\alpha - T_\beta$, then $C$ lies in some member of $\beta$; moreover, we may assume $x_\beta \in U$. If $x_\beta \neq x_\gamma$, then, since $r_\beta(x_\gamma) = x_\beta$, it follows that the component $C$ of $T_\gamma - T_\beta$ which contains $x_\gamma$ has $(x_\beta)$ for boundary, and hence $C \subset \text{Star}(U, \beta) \subset V$. Therefore the net $(x_\alpha)$ converges to $p$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Lemma 4. The function \( g: T_\infty \rightarrow X \) defined by \( g((x_\alpha)) = \lim(x_\alpha) \) is a continuous surjection.

Proof. Let \( p = \lim(x_\alpha) \) and suppose \( V \) is an open set containing \( p \). Choose a finite open cover \( \beta \) of \( X \) and \( T_\beta \in \mathcal{T} \) as in Lemma 3. If \( p \in U \in \beta \), let \( W = \pi_\beta^{-1}(U \cap T_\beta) \cap T_\infty \), a neighborhood of \((x_\alpha)\) in \( T_\infty \) (\( \pi_\beta \) denotes the projection function). If \((y_\gamma) \in W\) then \( y_\gamma \in U \) and hence, if \( T_\beta \subset T_\gamma \subset \mathcal{T} \), it follows that \( y_\gamma \in \text{Star}(U, \beta) \subset V \). Therefore \( g((y_\gamma)) \in \overline{V} \) and so \( g \) is continuous.

To see that \( g \) is surjective let \((x_\alpha) \in T_\infty \) with \((x_\alpha)\) eventually constant. That is, there exists \( T_\beta \in \mathcal{T} \) such that \( x_\gamma = x_\beta \) for all \( T_\gamma \in \mathcal{T} \) with \( T_\beta \subset T_\gamma \). Then \( g((x_\alpha)) = x_\beta \) and hence \( g(T_\infty) \supseteq \overline{\mathcal{T}} \). Since \( g \) is continuous and \( \bigcup \mathcal{T} \) is dense in \( X \) it follows that \( g(T_\infty) = X \).

Proof of Theorem 1. By [2] and Lemma 1 there is an arc \( L \) and a continuous surjection \( f: L \rightarrow T_\infty \). By Lemma 4 the function \( \varphi = gf: L \rightarrow X \) is the desired mapping.

Recently E. D. Tymchatyn [14] has applied Theorem 1 to prove that each finitely Suslinian Hausdorff continuum is the continuous image of an arc. This generalizes the result of Cornette, Pearson and the author [2], [10], [11], [17] for trees and rim-finite continua.

It is irresistible to inquire whether the condition of being approximated by finite trees is necessary as well as sufficient for a continuum to be the continuous image of an arc. I conjecture that the answer is affirmative.

3. The classical Hahn-Mazurkiewicz theorem. Recall that a dendrite is a metrizable tree. In attempting to deduce the classical theorem from Theorem 1, we consider a metric continuum \( M \). We wish to show that if \( M \) can be approximated by a sequence of finite dendrites then \( M \) is the continuous image of \([0, 1]\). It follows from Theorem 1 that \( M \) is the image of some arc, but we have no assurance that the arc is separable. The proof that \( M \) is the continuous image of \([0, 1]\) is facilitated by the following two lemmas.

Lemma 5. If \( D \) is a finite dendrite then there exists a continuous surjection \( f: [0, 1] \rightarrow D \).

Proof. Since \( D \) has only a finite set \( \{e_1, \ldots, e_n\} \) of endpoints, \( n \geq 2 \), we may write \( D = A_2 \cup \cdots \cup A_n \) where \( A_2 = [e_1, e_2] \) is an arc and \( A_k = [d_k, e_k] \) is an arc irreducible between \((A_1 \cup \cdots \cup A_{k-1}) \) and 
\( e_k \) where \( 2 < k \leq n \). There is a homeomorphism \( f_2: [0, 1] \rightarrow A_2 \); suppose \( f_{k-1}: [0, 1] \rightarrow (A_1 \cup \cdots \cup A_{k-1}) \) is a continuous surjection with \( f_{k-1}(t) = d_k \). Without loss of generality we may assume \( 0 < t < 1 \).

Define
\[
    h_1: [0, t] \rightarrow [0, \frac{1}{4}] \quad \text{by} \quad h_1(x) = x/4t,
\]
\[
    h_2: [t, 1] \rightarrow [\frac{3}{4}, 1] \quad \text{by} \quad h_2(x) = (x + 3 - 4t)/4(1 - t).
\]

Let
\[
    g_1: [\frac{1}{4}, \frac{1}{2}] \rightarrow [d_k, e_k] \quad \text{and} \quad g_2: [\frac{1}{2}, \frac{3}{4}] \rightarrow [e_k, d_k]
\]
be homeomorphisms which preserve the indicated endpoints. If we define
Lemma 6. If $D$ and $D'$ are finite dendrites with $D \subseteq D'$, $r: D' \to D$ is the natural monotone retraction and $f: [0, 1] \to D$ is a continuous surjection, then there exists a monotone mapping $s: [0, 1] \to [0, 1]$ and a continuous surjection $f': [0, 1] \to D'$ such that $fs = rf'$.

Proof. There are only finitely many elements $x_1, \ldots, x_n$ of $D$ which are the boundaries of components of $D' - D$. For each $i = 1, \ldots, n$ let

$$K_i = \{ x_i \} \cup \bigcup \{ C: C \text{ is a component of } D' - D \text{ and } x_i \in \overline{C} \},$$

and choose $t_i \in f^{-1}(x_i)$. Without loss of generality we assume $0 < t_1 < t_2 < \cdots < t_n < 1$. Define linear homeomorphisms $h_0, \ldots, h_n$ as follows:

$$h_0: [0, t_1] \to [0, 1/(2n + 1)] \quad \text{by} \quad h_0(x) = x/(2n + 1)t_1,$$

$$h_k: [t_k, t_{k+1}] \to [2k/(2n + 1), (2k + 1)/(2n + 1)]$$

by

$$h_k(x) = (x + 2kt_{k+1} - (2k + 1)t_k)/(2n + 1)(t_{k+1} - t_k),$$

$$k = 1, \ldots, n - 1,$$

$$h_n: [t_n, 1] \to \left[ \frac{2n}{2n + 1}, 1 \right] \quad \text{by} \quad h_n(x) = \frac{x + 2n - (2n + 1)t_n}{(2n + 1)(1 - t_n)}.$$ Each of the sets $K_i$ is a finite dendrite, so by Lemma 5 there is a continuous surjection

$$g_i: \left[ \frac{(2i - 1)}{(2n + 1)}, \frac{2i}{(2n + 1)} \right] \to K_i, \quad i = 1, \ldots, n.$$ Define $s: [0, 1] \to [0, 1]$ by

$$s = h_i^{-1} \quad \text{on} \quad \left[ \frac{(2i - 2)}{(2n + 1)}, \frac{(2i - 1)}{(2n + 1)} \right], \quad 1 \leq i \leq n + 1,$$

$$s(t) = t_i \quad \text{if} \quad t \in \left[ \frac{(2i - 1)}{(2n + 1)}, \frac{2i}{(2n + 1)} \right], \quad 1 \leq i \leq n,$$

and define $f': [0, 1] \to D'$ by

$$f' = \begin{cases} h_i^{-1} & \text{on} \quad \left[ \frac{(2i - 2)}{(2n + 1)}, \frac{(2i - 1)}{(2n + 1)} \right], \\ g_i & \text{on} \quad \left[ \frac{(2i - 1)}{(2n + 1)}, \frac{2i(2n + 1)}{(2n + 1)} \right], \quad 1 \leq i \leq n \end{cases}.$$ Then it is obvious that $s$ is continuous and monotone, that $f'$ is a continuous surjection and that $fs = rf'$.

We say that a metric continuum $M$ can be approximated by a sequence of finite dendrites if there exists a sequence $D_1, D_2, \ldots, D_n, \ldots$ of finite dendrites such that
(1) \( D_1 \subset D_2 \subset \cdots \subset D_n \subset \ldots \),
(2) \( \bigcup \{ D_n : n = 1, 2, \ldots \} \) is dense in \( M \),
(3) if \( C \) is a component of \( D_{n+1} - D_n \) then \( \text{diam}(C) < 2^{-n} \).

**Theorem 2.** If \( M \) is a metric continuum then the following statements are equivalent:

(i) there exists a continuous surjection \( \psi : [0, 1] \to M \),
(ii) \( M \) is a Peano continuum,
(iii) \( M \) can be approximated by a sequence of finite dendrites.

**Proof.** It is well known that (i) \( \Rightarrow \) (ii). (For example, consult [5].)

To see that (ii) \( \Rightarrow \) (iii), it is a consequence of the fact that \( M \) is compact and locally connected that \( M \) admits a sequence \( \mathcal{U}_n \) of finite connected open covers such that \( \mathcal{U}_{n+1} \) refines \( \mathcal{U}_n \) and \( \text{diam}(U) < 2^{-n} \) for each \( U \in \mathcal{U}_n \). Independent of the Hahn-Mazurkiewicz theorem it can be shown that each member of \( \mathcal{U}_n \) is arcwise connected. (See [18, Chapter II, §5, under the second remark on p. 39, together with 5.3].) Therefore it is possible to construct a sequence of finite dendrites \( D_1, D_2, \ldots \) such that \( D_n \) meets each member of \( \mathcal{U}_n \), \( D_n \subset D_{n+1} \), and each component of \( D_{n+1} - D_n \) lies in some member of \( \mathcal{U}_n \).

To prove (iii) \( \Rightarrow \) (i), let \( M \) be approximated by the sequence \( D_1 \subset D_2 \subset \ldots \) of finite dendrites. By Lemmas 5 and 6 there are continuous surjections \( f_n \) and continuous monotone surjections \( r_n \) and \( s_n \) so that the ladder

\[
\begin{align*}
D_1 & \xleftarrow{r_1} D_2 \xleftarrow{r_2} \cdots \xleftarrow{r_{n-1}} D_n \xleftarrow{r_n} \\
& \downarrow f_1 \quad \downarrow f_2 \quad \cdots \quad \downarrow f_n \\
[0, 1] & \xleftarrow{s_1} [0, 1] \xleftarrow{s_2} \cdots \xleftarrow{s_{n-1}} [0, 1] \xleftarrow{s_n} \cdots
\end{align*}
\]

is commutative. It follows that \( D_\infty = \text{inv lim} \{ D_n, r_n \} \) is a dendrite, the limit of the inverse sequence \( \{ [0, 1], s_n \} \) is \( [0, 1] \), and there is induced a continuous surjection \( f : [0, 1] \to D_\infty \). Let \( \psi = gf : [0, 1] \to M \).

**References**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OREGON 97403