PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 58, July 1976

HOMOTOPY EQUIVALENCES IN EQUIVARIANT TOPOLOGY

MARTIN FUCHS

ABSTRACT. Homomorphisms up to homotopy (higher homotopies that is) are generalized for the equivariant category. Homotopy equivalences have an inverse in this new category.

Introduction. In equivariant topology the notion of a homotopy equivalence presents a problem. Strictly within the equivariant category, homotopy equivalence seems to be too limited a concept,[1] e.g. the unit interval I (acting on itself as an H-space) is not of the same equivariant homotopy type as $\{1\}$ as a $\{1\}$-space.

Some of the rather general tools used in homotopy theory of topological groups and H-spaces (e.g. studying classifying spaces) can also be used in equivariant homotopy theory. In this paper we use G_∞-maps between G-spaces (as defined in 1.4, and similar to H_∞-maps between H-spaces) to study a new notion of homotopy equivalence. Roughly speaking, if X and $\overline{X}$ are G-spaces and if $f\colon X \to \overline{X}$ is a G-equivariant map and also an ordinary homotopy equivalence, then there exists a sequence of maps $g_n\colon \overline{X} \times (I \times G)^n \to X$ forming a G_∞-map such that f and the maps $\{g_n\}$ form a pair of G_∞-homotopy equivalences.

The complete theorem is stated in §2. The proof includes the proof of the corresponding theorem on H-spaces as stated in [2, Theorem 4.1] or in [1] as Proposition 1.17. We hope to convince the reader that this proof is not as messy as it is described by the authors of [1] on p. 13.

1. **Definitions.**

DEFINITION 1.1. An H-space G is a topological space with a continuous multiplication μ. We assume that μ is strictly associative. No unit element is needed.

DEFINITION 1.2. A topological space X is called a G-space, if G acts on X in a continuous and in an associative manner.

Multiplication and actions will be denoted by the usual juxtaposition.

DEFINITION 1.3. An H_∞-map $\underline{h}$ from G to $\overline{G}$ of length r[2] is a sequence of continuous maps $h_n\colon G \times (I_r \times G)^n \to \overline{G}$ such that

Presented to the Society, August 22, 1975; received by the editors August 13, 1974 and, in revised form, November 4, 1975.

AMS (MOS) subject classifications (1970). Primary 55D45, 55D10; Secondary 57E99.

[1] C. N. Lee, A. Wasserman (see [5]), and T. Petrie have relatively simple examples of pairs of manifolds X and Y with S^1-actions and an S^1-map $f\colon X \to Y$ such that f is an ordinary homotopy equivalence. But there is no equivariant map $g\colon Y \to X$ which is a homotopy equivalence.

[2] We will mention the length of maps only if it is essential to the context.

$$h_n(g_0, t_1, \ldots, t_n, g_n) = \begin{cases} h_{n-1}(g_0, t_1, \ldots, g_{i-1}g_i, \ldots, t_n, g_n), & t_i = 0, \\ h_{i-1}(g_0, t_1, \ldots, g_{i-1})h_{n-i}(g_i, \ldots, g_n), & t_i = r, \end{cases} \quad n \geqslant 0,$$

where $g_0, \ldots, g_n \in G$ and $t_1, \ldots, t_n \in I_r = [0, r] \subset \mathbf{R}$. If $r = 0$, the map h_0 is an ordinary homomorphism.

DEFINITION 1.4. Let X be a G-space, $\overline{X}$ be a $\overline{G}$-space and let $\underline{h}$ be an H_∞-map from G to $\overline{G}$ of length r. A G_∞-map $\underline{f}$ from X to $\overline{X}$ of length r associated to $\underline{h}$ is a sequence of maps $f_n\colon X \times (I_r \times G)^n \to \overline{X}$ such that

$$f_n(x, t_1, g_1, \ldots, t_n, g_n) = \begin{cases} f_{n-1}(x, t_1, \ldots, g_{i-1}g_i, \ldots, t_n, g_n), & t_i = 0, \\ f_{i-1}(x, t_1, g_1, \ldots, g_{i-1})h_{n-i}(g_i, \ldots, t_n, g_n), & t_i = r, \end{cases} \quad n \geqslant 0.$$

If $t_1 = 0$ we want of course

$$f_n(x, 0, g_1, t_2, \ldots, g_n) = f_{n-1}(xg_1, t_2, \ldots, g_n).$$

So G_∞-maps formally differ from H_∞-maps only in the first coordinate.

Composition of G_∞-maps is defined the same way as for H_∞-maps. If $n = 1$

$$(\bar{f} \circ f)_1(x, t_1, g_1) = \begin{cases} \bar{f}_0(f_1(x, t_1, g_1)), & 0 \leqslant t_1 \leqslant r, \\ \bar{f}_1(f_0(x), t_1 - r, h_0(y)), & r \leqslant t_1 \leqslant r + s, \end{cases}$$

which is the standard composition of homotopies. If $n > 1$ we have to form the composite of 2^n maps, each of which is defined on one of the 2^n rectangular boxes obtained from I_{r+s}^n be partitioning it with the hyperplanes $t_i = r$.

DEFINITION 1.5. Let X_1 be a G_1-space, X_2 a G_2-space, and X_3 a G_3-space. Let $\underline{h}$ be an H_∞-map of length r from G_1 and G_2 and $\underline{\bar{h}}$ an H_∞-map of length s from G_2 to G_3. Also let $\underline{f}$ be a G_∞-map of length r from X_1 to X_2 associated to $\underline{h}$, and let $\underline{\bar{f}}$ be a G_∞-map of length s from X_2 to X_3 associated to $\underline{\bar{h}}$. We define $(\bar{f}f)_n\colon X_1 \times (I_{r+s} \times G_1)^n \to X_3$ by

$$(\bar{f} \circ f)_n(x, t_1, g_1, \ldots, t_n, g_n) = \bar{f}_j(f_{i_1}(x, t_1, g_1, \ldots, t_{i_1}, g_{i_1}), t_{i_1+1} - r, h_{i_2 - i_1 - 1}(\cdots), \ldots,$$
$$t_{i_j+1} - r, h_{n-i_j-1}(g_{i_j+1}, \ldots, t_n, g_n)).$$

Here $x \in X_1$, $g_i \in G_1$, and for every sequence of natural numbers $0 \leqslant i_1 < \cdots < i_j < n$ $(j = 1, \ldots, n)$ we have $t_{i_1+1}, \ldots, t_{i_j+1} \in [r, r+s]$, while $t_i \in [0, r]$ whenever $i - 1 \neq i_1, \ldots, i_j$.

Composition formulas for H_∞-maps can also be found in [2], [3] and [4].

We leave it to the reader to verify that the maps $\{(\bar{f}f)_n\}$ form a G_∞-maps associated to $\{(\bar{h}h)_n\}$. The composition of G_∞-maps is associative and together with the G-spaces they form a category.

DEFINITION 1.6. Let $\underline{h}^0$ and $\underline{h}^1$ be H_∞-maps from G to $\overline{G}$, and let $\underline{f}^0$ and $\underline{f}^1$ be G_∞-maps from X to $\overline{X}$ associated to $\underline{h}^0$ and $\underline{h}^1$ respectively. If $\underline{h}^t$ $(0 \leqslant t \leqslant 1)$ is a family of H_∞-maps constituting an H_∞-homotopy between

$\underline{h}^0$ and $\underline{h}^1$, then a family $\underline{f}^t$ $(0 \leqslant t \leqslant 1)$ of G_∞-maps from X to $\overline{X}$ associated with $\underline{h}^t$ is called a G_∞-homotopy between $\underline{f}^0$ and $\underline{f}^1$.

DEFINITION 1.7. A G_∞-map $\underline{f}$ from X to $\overline{X}$ associated with the H_∞-map $\underline{h}$ from G to $\overline{G}$ is called a G_∞-homotopy equivalence if there exists an H_∞-map $\overline{\underline{h}}$ from $\overline{G}$ to G and a G_∞-map $\overline{\underline{f}}$ from $\overline{X}$ to X such that the respective compositions are G_∞-homotopic (and H_∞-homotopic respectively) to id_X and $\mathrm{id}_{\overline{X}}$ associated to id_G and $\mathrm{id}_{\overline{G}}$ respectively. The G_∞-map $\overline{\underline{f}}$ associated with $\overline{\underline{h}}$ is called a homotopy inverse of $\underline{f}$ associated with $\underline{h}$.

2. Theorems and proofs.

THEOREM. *Let X be a G-space and $\overline{X}$ be a $\overline{G}$-space. If $\underline{h}$ is an H_∞-map from G to $\overline{G}$ such that h_0 is an ordinary homotopy equivalence, and if $\underline{f}$ is a G_∞-map from X to $\overline{X}$ associated with $\underline{h}$ such that f_0 is an ordinary homotopy equivalence, then $\underline{f}$ is a G_∞-homotopy equivalence.*

REMARK. We construct both an inverse $\overline{\underline{h}}$ to $\underline{h}$ and an inverse $\overline{\underline{f}}$ to $\underline{f}$ associated with $\overline{\underline{h}}$. However, the construction of $\overline{\underline{f}}$ works for any inverse of $\underline{h}$, and also for every homotopy inverse of f_0.

Two special cases are of importance.

COROLLARY. *If X and $\overline{X}$ are G-spaces and $f\colon X \to \overline{X}$ is a G-equivariant map as well as an ordinary homotopy equivalence, then f is a G_∞-homotopy equivalence associated to 1_G.*

COROLLARY. *If H and $\overline{H}$ are H-spaces and $h\colon H \to \overline{H}$ is a strict homomorphism (or an H_∞-map) as well as an ordinary homotopy equivalence, then h is an H_∞-homotopy equivalence (see [2, Theorem 4.1] and [1]).*

The second corollary is obtained by choosing $X = H$ and $\overline{X} = \overline{H}$.

PROOF. We will construct an H_∞-map $\overline{\underline{h}}\colon \overline{G} \to G$ which is an H_∞-homotopy inverse to $\underline{h}$, and a G_∞-map $\overline{\underline{f}}\colon \overline{X} \to X$ associated to $\overline{\underline{h}}$ which is a G_∞-homotopy inverse to $\underline{f}$. The construction is by induction.

$n = 0$: Let $\bar{f}_0\colon \overline{X} \to X$ be an ordinary homotopy inverse to f_0 and $k_0\colon X \times I \to X$ be such that

$$k_0(x,t) = \begin{cases} x & \text{for } t = 0, \\ \bar{f}_0 f_0(x) & \text{for } t = 1, \end{cases} \qquad x \in X.$$

Then, since f_0 and $\bar{f}_0$ induce isomorphisms on the relevant homotopy classes of maps (compare e.g. [2, p. 205]) we can choose $\bar{k}_0\colon \overline{X} \times I \to \overline{X}$ such that

$$\bar{k}_0(\bar{x},t) = \begin{cases} \bar{x} & \text{for } t = 0, \\ f_0 \bar{f}_0(\bar{x}) & \text{for } t = 1 \end{cases}$$

in such a manner that there exists a map $u_0\colon X \times I^2 \to \overline{X}$ with

$$\begin{aligned} u_0(x,0,s_2) &= f_0 k_0(x,s_2), \\ u_0(x,1,s_2) &= \bar{k}_0(f_0(x),s_2), \end{aligned} \qquad s_2 \in I,\ x \in X,$$

and

$$\left.\begin{aligned} u_0(x,s_1,1) &= f_0\bar{f}_0f_0(x) \\ u_0(x,s_1,0) &= f_0(x) \end{aligned}\right. \qquad \text{for all } s_1 \in I,\ x \in X.$$

We will call $\bar{f}_0$ together with $(k_0, \bar{k}_0)$ and u_0 a compatible homotopy inverse to f_0.

Similarly we choose a compatible homotopy inverse $\bar{h}_0$ together with homotopies $(l_0, \bar{l}_0)$ and v_0 for the map h_0 (compare [2, p. 205]). We notice that this second choice is independent of the choice of $\bar{f}_0$.

INDUCTION HYPOTHESIS. Assume we already constructed $(\bar{f}_0, \ldots, \bar{f}_{n-1})$ as the first n functions of a G_∞-homotopy inverse for $\underline{f}$ associated with $(\bar{h}_0, \ldots, \bar{h}_{n-1})$, the n functions of an H_∞-homotopy inverse of $\underline{h}$. Furthermore assume that we constructed these functions as compatible inverses to $(f_0, \ldots, f_{n-1})$ and $(h_0, \ldots, h_{n-1})$. For convenience sake we assume that all maps of $\underline{f}$ and $\underline{h}$ have length one. Also $\bar{f}_0, \ldots, \bar{f}_{n-1}$ and $\bar{h}_0, \ldots, \bar{h}_{n-1}$ are assumed to have length one. The homotopies $k_0, \ldots, k_{n-1}, \bar{k}_0, \ldots, \bar{k}_{n-1}, l_0, \ldots, l_{n-1}$ and $\bar{l}_0, \ldots, \bar{l}_{n-1}$ are to be of length two. The maps $u_0, \ldots, u_{n-1}$ and $v_0, \ldots, v_{n-1}$ are to have length three. These last maps shall satisfy the obvious boundary conditions as described in 1.3 and 1.4, in addition to the compatibility conditions.

Now we will construct $u_n\colon X \times (I_3 \times G)^n \times I^2 \to \bar{X}$. (The maps $\bar{f}_n$, k_n and $\bar{k}_n$ will be defined in the process of this construction.) For our construction by induction u_n shall have the following properties.

(1) $$u_n | X \times (I_3 \times G)^n \times I \times \{0\} = (\bar{e}_1 \circ f \circ e_1)_n \times \mathrm{id}_I .$$

Here e_1 and $\bar{e}_1$ stands for the identity morphism of X and $\bar{X}$ respectively with length 1. We observe that this part of u_n is already known.

(2) $$u_n | X \times (I_3 \times G)^n \times I \times \{1\} = (f \circ \bar{f} \circ f)_n \times \mathrm{id}_I .$$

The composition $(f \circ \bar{f} \circ f)_n$ contains the map $\bar{f}_n$ only if $1 \leqslant t_i \leqslant 2$ (for $i = 1, \ldots, n$). For these values of t_i we have

$$(f \circ \bar{f} \circ f)_n(x, t_1, g_1, \ldots, g_n) = f_0\bar{f}_n(f_0(x), t_1, h_0(g_1), \ldots, t_n, h_0(g_n)).$$

The rest of $(f \circ \bar{f} \circ f)_n$ uses $\bar{f}_k$ with $1 \leqslant k < n$. Because of this and the boundary properties of $\bar{f}_n$ we know this part of u_n except when $1 < t_i < 2$, $i = 1, \ldots, n$.

(3) $$u_n | X \times (I_3 \times G)^n \times \{0\} \times I = (\bar{e}_{1-s_2} \circ f \circ k_{1+s_2})_n .$$

$\bar{e}_{1-s_2}$ stands for the identity morphism of $\bar{X}$ with length $1 - s_2$, and k_{1+s_2} stands for the homotopy k with length $1 + s_2$, where s_2 is the second coordinate of $I \times I$. From the composition rule for G_∞-maps we see that this restriction of u_n is known except for $1 < t_i < 2$ when $s_2 = 1$, and $2 - s_2 < t_i < 3$ when $0 < s_2 < 1$ $(i = 1, \ldots, n)$, i.e. when

$$u_n(x, t_1, g_1, \ldots, t_n, g_n, 0, s_2) = \begin{cases} f_0\bar{f}_n(f_0(x), t_1, h_0(g_1), \ldots, t_n, h_n(g_n)), & s_2 = 1, \\ f_0 k_{1+s_2,n}(x, t_1, \ldots, t_n, g_n, s_2), & 0 < s_2 < 1. \end{cases}$$

(4) $$u_n | X \times (I_3 \times G)^n \times \{1\} \times I = (\bar{k}_{1+s_2} \circ f \circ e_{1-s_2})_n$$

which is known except for $1 < t_i < 2$ when $s_2 = 1$, and $0 < t_i < 1 + s_2$ when $0 < s_2 < 1$, i.e. when

$$u_n(x, t_1, g_1, \ldots, t_n, g_n, 1, s_2) = \begin{cases} f_0 \bar{f}_n(f_0(x), t_1, h_0(g_1), \ldots, t_n, h_0(g_n)), & s_2 = 1, \\ \bar{k}_{1+s_2,n}(f_0(x), t_1, h_0(g_1), \ldots, t_n, h_0(g_n)), & 0 < s_2 < 1. \end{cases}$$

(5) $u_n | X \times \partial(I_3 \times G)^n \times I^2$ is defined by $u_0, \ldots, u_{n-1}$ and $v_0, \ldots, v_{n-1}$ according to the properties of G_∞- and H_∞-maps as described in 1.3 and 1.4. Hence this part of u_n is known.

To find all of u_n, we are going to use the following consequence of the homotopy extension property:

LEMMA. *If $A \subset X$ has the homotopy extension property and if $r\colon Y_1 \rightleftarrows Y_2 \colon s$ are homotopy equivalences, then if $f_1\colon A \to Y_1$ is a map such that $r \circ f_1$ has an extension $g_2\colon X \to Y_2$, we know that f_1 has an extension $g_1\colon X \to Y_1$ by the homotopy extension property.*

We observe that the extension problem resulting from (3) is homeomorphic to extending a map, which is known on all of $Y \times \partial I^{n+1}$ with the exception of one face of $Y \times I^{n+1}$, to all of $Y \times I^{n+1}$. We extend, and use the lemma to obtain $K_{1+s_2}\colon X \times (I_{1+s_2} \times G)^n \times \{0\} \times I \to X$. For $s_2 = 1$ we see that K_{1+s_2} provides an extension of $\bar{f}_n \circ f_0$ from $X \times \partial([1,2] \times G)^n$ to $X \times ([1,2] \times G)^n$. To obtain the extension of $\partial \bar{f}_n\colon \bar{X} \times \partial([1,2] \times \bar{G})^n \to X$ we use the following

LEMMA. *Given a homotopy equivalence $a\colon A \to \bar{A}$ a map $\partial c\colon \bar{A} \times \partial I^n \to \bar{B}$, and a map $w\colon A \times I^n \to \bar{B}$ that extends $\partial c(a \times 1)$, then ∂c extends to a map $c\colon \bar{A} \times I^n \to \bar{B}$ such that $c(a \times 1)$ is homotopic to w keeping the boundary fixed.*

Since on $X \times \partial([1,2] \times G)^n \times \{1\} \times \{1\}$ we have

$$K_{1+s_2}(x, t_1, g_1, \ldots, t_n, g_n, 1, 1) = \bar{f}_n(f_0(x), t_1, h_0(g_1), \ldots, t_n, h_0(g_n)),$$

we obtain that $K_{1+s_2} | X \times ([1,2] \times G)^n \times \{1\} \times \{1\}$ is homotopic to $\bar{f}_n \circ f_0 \times (1 \times h_0)^n$ relative to the boundary $X \times \partial([1,2] \times G)^n \times \{1\} \times \{1\}$. We combine K_{1+s_2} with such a homotopy to obtain k_{1+s_2}, with $k_{1+s_2} | X \times ([1,2] \times G)^n \times \{1\} \times \{1\} = \bar{f}_n \circ f_0 \times (1 \times h_0)^n$.

We now know u_n except on $X \times (I_3 \times G)^n \times \mathring{I}^2$ and the part of $X \times (I_3 \times G)^n \times \{1\} \times I$ with $0 < t_i < 1 + s_2$ and $0 < s_2 < 1$. This extension problem is homeomorphic to the problem of extending a map which is known on all of $Y \times \partial I^{n+2}$ with the exception of one face to all of $Y \times I^{n+2}$. Let U_n be such an extension. We have to use the second lemma once more to obtain $\bar{k}_{1+s_2}$. Again $\bar{k}_{1+s_2,n}(f_0 \times 1_I \times h_0 \times \cdots \times h_0)$ is homotopic to the right restriction of U_n leaving the boundary fixed. We alter U_n by this homotopy and obtain u_n.

The construction of v_n, $\bar{h}_n$, l_n, and $\bar{l}_n$ is completely analogous and is left to the reader.

References

1. J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Math., vol. 347, Springer-Verlag, Berlin and New York, 1973.

2. M. Fuchs, *Verallgemeinerte Homotopie-Homomorphismen und klassifizierende Räume*, Math. Ann. **16** (1965), 197–230. MR **33** #3295.

3. A. Clark, *Homotopy commutativity and the Moore spectral sequence*, Pacific J. Math. **15** (1965), 65–74. MR **31** #1679.

4. B. Drachman, *A generalization of the Steenrod classification theorem to H-spaces*, Trans. Amer. Math. Soc. **153** (1971), 53–88. MR **44** #5961.

5. C. N. Lee and A. G. Wasserman, *On the groups* $JO(G)^*$, University of Michigan (to appear).

Department of Mathematics, Michigan State University, East Lansing, Michigan 48824