3-MANIFOLDS FIBERING OVER $S^1$

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Abstract. Let $M$ be a closed 3-manifold that is the total space of a fiber bundle with base $S^1$ and fiber the closed 2-manifold $F$. Assume that genus $(F) \geq 2$ if $F$ is orientable, and that genus $(F) \geq 3$ if $F$ is nonorientable. We say that $M$ has unique fiber over $S^1$ if, for any fibering of $M$ over $S^1$ with fiber $F'$, we have $F' \cong F$. We prove that $M$ has unique fiber over $S^1$ if and only if rank $(H_1(M;\mathbb{Z})) = 1$. In the case that rank $(H_1(M;\mathbb{Z})) \neq 1$, $M$ fibers over $S^1$ with fiber any of infinitely many distinct closed surfaces.

In [5], Tollefson proved that if $M$ is a 3-manifold of the form $F \times S^1$, where $F$ is a closed oriented surface of genus $g \geq 2$, then $M$ fibers over $S^1$ with fiber any of infinitely many distinct surfaces. We extend this result to a characterization of uniqueness of the fiber in 3-manifolds fibering over $S^1$.

All manifolds and maps considered will be differentiable (say $C^1$). We say that the closed 3-manifold $M$ fibers over $S^1$ with fiber $F$ if $M$ is diffeomorphic to the quotient manifold $(F \times I)/d$ obtained by identifying $F \times \{0\} \subseteq F \times I$ with $F \times \{1\}$ under the diffeomorphism $d: F \to F$. We say that the fiber is unique if in any other fibering $M \cong (F' \times I)/d'$ we have $F' \cong F$. Our result is then:

Theorem. Suppose $M$ is a closed 3-manifold that fibers over $S^1$ with fiber a surface $F$ of genus $g \geq 2$ ($g \geq 3$ if $F$ is nonorientable); then the fiber is unique if and only if rank $(H_1(M;\mathbb{Z})) = 1$.

Proof. To prove the sufficiency of the rank condition suppose that $M$ fibers as $(F \times I)/d$ and as $(F' \times I)/d'$, with $F' \cong F$. Note that $\pi_1(M)$ decomposes as a semidirect product $\pi_1(F) \rtimes d_* \mathbb{Z}$ (the action of a generator $z$ of the infinite cyclic factor on the normal subgroup $\pi_1(F)$ is given by $zfz^{-1} = d_*(f)$, for all $f \in \pi_1(F)$) and also as $\pi_1(F') \rtimes d_* \mathbb{Z}$. Now assume that rank $(H_1(M;\mathbb{Z})) = 1$. Then, since $z$ maps to the generator of an infinite cyclic direct summand of $H_1(M;\mathbb{Z})$ under the abelianizing (Hurewicz) homomorphism, one can easily prove that $\pi_1(F) \subseteq \pi_1(M)$ consists of just those elements that are torsion modulo the commutator subgroup of $\pi_1(M)$. The same holds for $\pi_1(F') \subseteq \pi_1(M)$, and hence we see that $\pi_1(F)$ is isomorphic to $\pi_1(F')$. This contradiction shows that we must have rank $(H_1(M;\mathbb{Z})) \geq 2$.

To prove the converse we assume that $M = (F \times I)/d$ and that rank $(H_1(M;\mathbb{Z})) \geq 2$. To find a fiber $F_n \subseteq M$ distinct from $F$ we first construct a map $P: M \to S^1 \times S^1$; $F_n$ will then be realized as the inverse

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image of a 1-submanifold $W_\eta \subseteq S^1 \times S^1$ on which $P$ is transverse.

We first show that there is an epimorphism $\pi : \pi_1(F) \to \mathbb{Z}$ satisfying $\pi d_\ast^{-1} = \pi$. In the commutative diagram, $\beta$ is the projection with kernel $\pi_1(F)$, $\alpha$ is the Hurewicz homomorphism, and $\beta$ is induced by $\beta$.

$$
\begin{array}{cccc}
\pi_1(F) & \xrightarrow{\gamma = \alpha|_{\pi_1(F)}} & \ker \beta & \xrightarrow{} \mathbb{Z} \\
\downarrow & & \downarrow & \\
\pi_1(F) \times_d \mathbb{Z} & \xrightarrow{\alpha} & H_1(M; \mathbb{Z}) & \xrightarrow{} \mathbb{Z} \\
\downarrow & & \downarrow & \\
\mathbb{Z} & \xrightarrow{l} & \mathbb{Z} & \\
\end{array}
$$

Note that $\gamma = \alpha|_{\pi_1(F)}$ is onto. Also, we may choose a generator $z \in \pi_1(M)$ of the infinite cyclic factor so that $d_\ast^{-1}(x) = z^{-1}xz$, for all $x \in \pi_1(F)$; hence

$$
\gamma d_\ast^{-1}(x) = \gamma(z^{-1}xz) = \alpha(z^{-1}xz) = \alpha(x) = \gamma(x).
$$

Since $\ker \beta$ has at least one infinite cyclic summand by our assumption on $H_1(M; \mathbb{Z})$, we may define $\pi$ to be the composition of $\gamma$ with the projection onto such a summand.

Next, there is a differentiable map $p : F \to S^1$ realizing $\pi$; i.e., with $p_\ast = \pi$. By Lemma 3.2 of Jaco [2] we may assume that $p$ is transversal with respect to a point $s_0 \in S^1$, and that, for each $s$ in some open interval about $s_0$, $p^{-1}(s)$ is a simple closed curve in $F$. Since $p_\ast d_\ast^{-1} = p_\ast$, there is a homotopy $p_t (t \in [0,1])$ of $p$ to $pd^{-1}$; we may assume that this homotopy defines a differentiable map $F \times I \to S^1$ (cf. Hu [1, Lemma 2]). We will also assume that $p_t = p$, for $t \in [0,1/3]$ say, and that $p_t = pd^{-1}$, for $t \in [2/3,1]$. It follows that we may define a differentiable map

$$
P : (F \times I)/d \to S^1 \times S^1 \quad (= (S^1 \times I)/1)
$$

by $P(x,t) = (p_t(x),t)$.

We now want to find a 1-submanifold $W_\eta \subseteq S^1 \times S^1$ so that $P$ is transverse on $W_\eta$. We consider local coordinates of the form

$$(x_1, x_2, t) \in (F \times I)/d$$

(whence $x = (x_1, x_2)$ are local coordinates on $F$ and $t \in J(\text{open}) \subseteq I$) and $(s, t) \in S^1 \times S^1$. Then the derivative $DP$ of $P$ is represented by the matrix

$$
\left[
\begin{array}{ccc}
\frac{\partial p_t}{\partial x_1} & \frac{\partial p_t}{\partial x_2} & \frac{\partial p_t}{\partial t} \\
0 & 0 & 1
\end{array}
\right]
$$

with respect to the bases $\{\partial/\partial x_1, \partial/\partial x_2, \partial/\partial t\} \subseteq T(M)_{(x,t)}$ and $\{\partial/\partial s, \partial/\partial t\} \subseteq T(S^1 \times S^1)_{p(x,t)}$. Thus the image of the tangent vector $(0,0,1)$ (i.e., $\partial/\partial t$) in $T(M)_{(x,t)}$ is $(\partial p_t/\partial t,1) \in T(S^1 \times S^1)_{p(x,t)}$. We want to choose $W_\eta \subseteq S^1 \times S^1$ so that, at $w \in W_\eta$, $T(W_\eta)_w$ does not contain any vector of the form $DP_{(x,t)}(\partial/\partial t)$ ($(x,t) \in P^{-1}(w)$). If we fix finite covers of $M$ and
$S^1 \times S^1$ consisting of charts of the above form, then

$$m = \sup_M |\partial p_i/\partial t|$$

(supremum over all local representations of $p_i$ in terms of these fixed charts) will be finite. A sufficient condition that $P$ be transverse on $W_n$ is that if $a \partial /\partial s + b \partial /\partial t \in T(W_n)$, then $0 < b/a < 1/m$. It is clear that, for all sufficiently large $n$, there is such a simple closed curve $W_n$ that winds $n$ times around the $S^1 \times \{0\}$ factor of $S^1 \times S^1$ and once around the $\{s_0\} \times S^1$ factor.

Fix such a $W_n$ and define $F_n \subseteq M$ to be the component of $P^{-1}(W_n)$ that contains $p^{-1}(s_0)$. Then $F_n$ is a codimension one (differentiable) submanifold of $M$ (cf. [3, pp. 21-25]).

To see that $M$ fibers as $(F_n \times I)/\partial_n$, we use the fact that $F_n$ is a section in the sense of Smale ([2, p. 99]) for the "natural" flow $\phi$ on $M$ determined by the diffeomorphism $d$. The flow $\phi : M \times \mathbb{R}^1 \to M$ is defined as follows: $M$ may be realized as the quotient space of $F \times \mathbb{R}^1$ under the equivalence relation $(x, t) \sim (dx, t + 1)$; $\phi$ is then the flow induced by the constant vector field $(0, 0, 1)$ on $F \times \mathbb{R}^1$. The fact that $DP(0,0,1)$ is not in $T(W_n)$ implies that $F_n$ is transverse to $\phi$. To show that $F_n$ is a (complete) section for $\phi$ it remains to show that, for any point $(x, t) \in F_n$, there is a time $t > 0$ so that $\phi(x, t, t') \in F_n$ (cf. remark following the definition of section, p. 99 of [4]). To see this, let $s_{0,0} = (s_0,0), s_{0,1} = (s_0,t_1), \ldots, s_{0,n-1} = (s_0,t_{n-1})$ denote the successive intersections of $W_n$ with $\{s_0\} \times S^1$, let $(s_{0,0},s_{0,1})$ denote the half-open segment of $W_n$ running from $s_{0,0}$ to $s_{0,1}$, and let $F_{n,1}$ denote $P^{-1}((s_{0,0},s_{0,1}))$. We may assume that $((s_{0,0},s_{0,1}) \subseteq S^1 \times [0,1/3]$; in this case Int $(F_{n,1})$ is mapped homeomorphically onto $F - (p^{-1}(s_0))$ under the projection of $F \times I$ onto the first factor. Thus we may define a (height) function $h : F \to I$ by $h(x) = t$ if $(x, t) \in F_{n,1}$ ($h$ is continuous except at $p^{-1}(s_0)$). Then if $(x, t) \in F_n$ we have $1 - t + h(d^{-1}x) > 0$ and $\phi(x, t, 1 - t + h(d^{-1}x)) \in F_n$ as required. It now follows by Theorem 2.2 of [4] that $M \cong (F_n \times I)/\partial_n$, where $\partial_n$ is the diffeomorphism of $F_n$ induced by $\phi$.

To complete the proof we must show that $F_n \cong F$. We show, in fact, that we can obtain surfaces $F_n$ of arbitrarily high genus by varying $n$. Suppose that $[s_{0,0},s_{0,1}], \ldots, [s_{0,k-1},s_{0,k}]$ (notation as above) are all contained in $S^1 \times [0,1/3]$, and let

$$F_{n,i} = P^{-1}([s_{0,i-1},s_{0,i}]), \quad i = 1, 2, \ldots, k,$$

$$F_{n,0} = P^{-1}(\text{cl } (W_n - [s_{0,0},s_{0,k}])).$$

Then each $F_{n,i}$ is a compact surface with two boundary components and, for $i = 1, 2, \ldots, k$, we have $F_{n,i} \cong F - N(p^{-1}(s_0))$ (here $N(p^{-1}(s_0))$ denotes a regular neighborhood of $p^{-1}(s_0)$). Thus

$$\chi(F_{n,0}) \leq 0,$$

$$\chi(F_{n,i}) = 2 - 2g \quad (i = 1, 2, \ldots, k, F \text{ orientable}) \text{ or }$$

$$\chi(F_{n,i}) \leq 2 - g \quad (i = 1, 2, \ldots, k, F \text{ nonorientable}).$$
It follows that $\chi(F_n) \leq k(2 - 2g)$ ($\chi(F_n) \leq k(2 - g)$ in case $F$ is nonorientable) is arbitrarily large negative, as asserted.

**Bibliography**


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