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STRONGLY EXPOSED POINTS IN WEAKLY COMPACT CONVEX SETS IN BANACH SPACES

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ABSTRACT. A "purely geometric" proof of the Lindenstrauss-Troyanski result ([2], [6]) on strongly exposed points of weakly compact sets in Banach spaces is given.

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We consider a Banach-space X, $\| \ \|$. If $\xi \in X$ and if $\varepsilon > 0$, then $B(\xi, \varepsilon) = \{x \in X; \|x - \xi\| < \varepsilon\}$. The closed convex hull of a set $A \subset X$, is denoted $\bar{c}(A)$. Let C be a convex subset of X, then C^e is the set of the extremal points of C. If K is a convex, weakly compact, B a convex, closed, bounded set of X and J a closed subinterval of $[0, 1]$, (K, B, J) denotes the closed, convex set $\{(1 - t)k + tb; k \in K, b \in B, t \in J\}$.

We have to introduce a few geometrical definitions, referring to [4].

Suppose that C is a nonempty, bounded, closed and convex subset of X. Let $M(C) = \sup\{\|x\|; x \in C\}$. If $f \in X^*$ with $\|f\| = 1$, let $M(f, C) = \sup\{f(x); x \in C\}$, and for each $\alpha > 0$, let $S(f, \alpha, C) = \{x \in C; f(x) \geqslant M(f, C) - \alpha\}$. Such a set is called a "slice" of C. A point ξ of C is called "strongly exposed" if there exists $f \in X^*$ ($\|f\| = 1$) such that $\forall \varepsilon > 0$, $\exists \alpha > 0$ with $\xi \in S(f, \alpha, C) \subset B(\xi, \varepsilon)$. Let S be the set of all $f \in X^*$ such that $\|f\| = 1$ and f strongly exposes some point of C.

PROPOSITION 1. *If K is a nonempty, convex, weakly-compact subset of X and if B is convex, closed and bounded with $K \cap B = \varnothing$, then the set $D = \bar{c}(K \cup B) = (K, B, [0, 1])$ has the following property: $\forall \varepsilon > 0$, $\exists \xi \in K$ such that $\xi \notin \bar{c}(D \backslash B(\xi, \varepsilon))$.*

First we observe that it is sufficient to prove the proposition for X separable.

Indeed, suppose D does not have the required property. Then there exists $\varepsilon > 0$ with $\forall x \in K$: $x \in \bar{c}(D \backslash B(x, \varepsilon))$ and hence, $\forall x \in K$, $\exists A(x) \subset D$ with the properties: $x \in \bar{c}(A(x))$, $A(x) \cap B(x, \varepsilon) = \varnothing$ and $A(x)$ is countable. By induction we construct a sequence $(K_n, B_n)_n$ where K_n is countable in K and B_n is countable in B.

Let $(K_0, B_0) = (\{x\}, \varnothing)$, where x is some element of K. Suppose we already found (K_n, B_n).

Consider $\forall y \in \bigcup_{x \in K_n} A(x)$ an element k_y in K and b_y in B, with $y \in \bar{c}(\{k_y, b_y\})$.

Let

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$$K_{n+1} = \left\{ \sum_{i=1}^{m} \lambda_i x_i, m \in \mathbf{N}^*, x_i \in K_n \cup \bigcup_{x \in K_n} \{k_y; y \in A(x)\}, \right.$$
$$\left. \lambda_i \in \mathbf{Q}^+ (1 \leqslant i \leqslant m) \text{ and } \sum_{i=1}^{m} \lambda_i = 1 \right\},$$

and

$$B_{n+1} = \left\{ \sum_{i=1}^{m} \lambda_i x_i; m \in \mathbf{N}^*, x_i \in B_n \cup \bigcup_{x \in K_n} \{b_y; y \in A(x)\}, \right.$$
$$\left. \lambda_i \in \mathbf{Q}^+ (1 \leqslant i \leqslant m) \text{ and } \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

Then K_{n+1} and B_{n+1} are still countable, which completes the construction.

Consider then $K' = \overline{\bigcup_n K_n}$, $B' = \overline{\bigcup_n B_n}$, $D' = \bar{c}(K' \cup B')$ and the closed linear span X_0 of D'.

Since $\forall n$: $\bar{c}(K_n) \subset \overline{K}_{n+1}$, K' is convex and closed. The same holds for B'. X_0 is a separable Banach-space wherein K' is nonempty, convex, weakly-compact and B' convex, closed and bounded. We show that $D' \subset X_0$ does not have the property mentioned in the proposition. Choose $x' \in K'$ and take $0 < \delta < \varepsilon/2$. There exist $n \in \mathbf{N}$ and $x \in K_n$ with $\|x - x'\| \leqslant \delta$. We obtain:

$$x \in \bar{c}(A(x)) \subset \bar{c}\left(\bigcup_{y \in A(x)} \bar{c}(\{k_y, b_y\}) \backslash B(x, \varepsilon) \right) \subset \bar{c}(D' \backslash B(x, \varepsilon))$$
$$\subset \bar{c}(D' \backslash B(x', \varepsilon/2)).$$

Since this holds, $\forall \delta$ $(0 < \delta < \varepsilon/2)$, we have $x' \in \bar{c}(D' \backslash B(x', \varepsilon/2))$. D' does not have the property and we can restrict ourselves to the case of a separable Banach-space.

LEMMA. $D = \bar{c}(D^e \cup B)$.

PROOF. Clearly $E = \bar{c}(D^e \cup B) \subset D$. For the reverse inclusion it is sufficient to prove that $K \subset E$. Suppose that $x_0 \in K \backslash E$. Then $\exists u \in X^*$ such that $u(x_0) > \sup\{u(x); x \in D^e \cup B\} \geqslant \sup\{u(x); x \in B\}$. Let $\alpha = \sup\{u(x); x \in K\}$, and let $y \in D$. Then $y = \lambda x_1 + (1 - \lambda) x_2$ with $x_1 \in K$, $x_2 \in B$, $\lambda \in [0, 1]$. It follows that

$$u(y) = \lambda u(x_1) + (1 - \lambda) u(x_2) \leqslant \lambda u(x_1) + (1 - \lambda) u(x_0)$$
$$= u(\lambda x_1 + (1 - \lambda) x_0) \leqslant \alpha.$$

This shows that $\sup\{u(y); y \in D\} = \alpha$, and, since $u(x_2) < u(x_0) \leqslant \alpha$, $u(y) = \alpha$ implies that $\lambda = 1$ or $y \in K$. Let $D_1 = \{y \in D; u(y) = \alpha\}$. Then D_1 is a supporting set of D, and $D_1 \subset K$. Since D_1 is weakly compact, it contains an extreme point z. Then $z \in D^e$, but since $u(z) = \alpha$, $z \notin (D^e \cup B)$. This contradiction establishes that $K \subset E$ or $D = E$.

PROOF OF PROPOSITION 1. The proof is a modification of a proof in Namioka [3]. We assume that X is separable. By the lemma, $A = D^e \backslash B \neq \varnothing$. Since $A \subset K$ the weak closure $\overline{A}$ of A is a Baire space relative to the weak topology. Since X is separable, there is a sequence $\{x_n\}$ in X such that

$$\bar{A} \subset \bigcup_{n=1}^{\infty} \bar{B}(x_n, \varepsilon/4).$$

Note that each $\bar{B}(x_n, \varepsilon/4)$ is weakly closed. Hence there is a weakly open set N in X such that $N \cap \bar{A} \neq \varnothing$ and $N \cap \bar{A} \subset \bar{B}(x_n, \varepsilon/4)$ for some n. Then clearly diam $(N \cap \bar{A}) \leqslant \varepsilon/2$. Let

$$D_1 = \bar{c}(N \cap \bar{A}) \quad \text{and} \quad D_2 = \bar{c}((\bar{A} \backslash N) \cup B).$$

Then we observe that D_1 is weakly compact and diam $D_1 \leqslant \varepsilon/2$.

Since A is weakly dense in $\bar{A}$, $N \cap A \neq \varnothing$. Fix $\xi \in N \cap A \subset K$. Then $\xi \notin D_2$. For if $\xi \in D_2$, then $\xi \in \bar{c}((K \backslash N) \cup B) \subset (\bar{c}(K \backslash N), B, [0,1])$. Since $\xi \in D^e$ and $\xi \notin B$, we have $\xi \in \bar{c}(K \backslash N)$. It then follows from the Krein-Milman theorem that $\xi \in K \backslash N$, because $K \backslash N$ is weakly compact. This contradicts $\xi \in N \cap A$.

Because $D = \bar{c}(D^e \cup B)$, $D = \bar{c}(D_1 \cup D_2) = (D_1, D_2, [0,1])$. Let $C = (D_1, D_2, [\varepsilon/5d, 1])$, where $d =$ diam D. Then C is a closed convex subset of D.

If $\xi \in C$, then $\xi = (1-\lambda)x_1 + \lambda x_2$ where $x_1 \in D_1$, $x_2 \in D_2$ and $\lambda \in [\varepsilon/5d, 1]$. Since ξ is extreme, this implies that $\xi \in D_2$, which is impossible. Hence $\xi \notin C$. Let $y_1, y_2 \in D \backslash C$. Then $y_i = (1-\lambda_i)x_1^i + \lambda_i x_2^i$, where $x_1^i \in D_1$, $x_2^i \in D_2$ and $\lambda_i \in [0, \varepsilon/5d]$ $(i = 1,2)$. We then have:

$$\begin{aligned} \|y_1 - y_2\| &\leqslant \|x_1^1 - x_1^2\| + \lambda_1 \|x_1^1 - x_2^1\| + \lambda_2 \|x_1^2 - x_2^2\| \\ &\leqslant \varepsilon/2 + \varepsilon d/5d + \varepsilon d/5d = 9\varepsilon/10. \end{aligned}$$

Since $\xi \in D \backslash C$ and diam $(D \backslash C) < \varepsilon$, it follows that $D \backslash C \subset B(\xi, \varepsilon)$. Therefore $D \backslash B(\xi, \varepsilon) \subset C$ and $\bar{c}(D \backslash B(\xi, \varepsilon)) \subset C$. Thus $\xi \notin \bar{c}(D \backslash B(\xi, \varepsilon))$ and the proof is complete.

REMARK. The fact that X, $\| \ \|$ is complete is not used and the assertion is still true when X is only a normed space.

PROPOSITION 2. *Let C be a convex and weakly-compact subset of X. If $S(f, \alpha, C)$ is a slice of C, then $\forall \varepsilon > 0$: $\exists g \in X^*$, $\exists \beta > 0$ such that $S(g, \beta, C)$ is a slice of C with diameter $\leqslant \varepsilon$, $S(g, \beta, C) \subset S(f, \alpha, C)$ and $\|f - g\| \leqslant \varepsilon$.*

PROOF. The proof is a modification of a proof in Phelps [4]. By translation we may assume that $0 \in H = \{x \in X; f(x) = M(f, C) - \alpha\}$. Hence $H = f^{-1}(0)$ and $\alpha = M(f, C) > 0$. We may also assume that $\varepsilon < \min(1, \alpha)$. Choose λ so that $\lambda > 2M(C)/\varepsilon$, and let $K = S(f, \alpha/2, C)$ and $B = H \cap \bar{B}(0, \lambda)$. Since $K \cap B = \varnothing$ and $K \neq \varnothing$, we may apply Proposition 1 to K and B.

Let $D = \bar{c}(K \cup B)$. Then $\exists \xi \in K$ such that $\xi \notin \bar{c}(D \backslash B(\xi, \varepsilon/2))$. If $x \in B$, then $\|x - \xi\| \geqslant |f(x - \xi)| \geqslant \alpha/2 > \varepsilon/2$. Therefore $B \subset \bar{c}(D \backslash B(\xi, \varepsilon/2))$. By the separation theorem, $\exists g \in X^*$ such that $\|g\| = 1$ and $g(\xi) > \sup\{g(x); x \in D \backslash B(\xi, \varepsilon/2)\} \geqslant 0$. Since $M(g, C) \geqslant g(\xi)$, we may write $\sup\{g(x); x \in D \backslash B(\xi, \varepsilon/2)\} = M(g, C) - 2\beta$ with $\beta > 0$.

Then $S(g, \beta, C) \subset B(\xi, \varepsilon/2)$, and hence diam $S(g, \beta, C) \leqslant \varepsilon$. Suppose that $f(x) = 0$ and $\|x\| \leqslant 1$. Then $\lambda x \in B$, and hence $\lambda g(x) < g(\xi)$ or $g(x) < \lambda^{-1} g(\xi)$. By Lemma 2 of [1], this implies that either $\|f - g\| \leqslant 2\lambda^{-1} g(\xi)$

$\leqslant 2\lambda^{-1}M(C) < \varepsilon$ or $\|f+g\| \leqslant 2\lambda^{-1}g(\xi)$. If the second possibility occurs, then $g(\xi) < (f+g)(\xi) \leqslant \|f+g\|\,\|\xi\| \leqslant 2\lambda^{-1}g(\xi)M(C) < g(\xi)\varepsilon$, which implies that $1 < \varepsilon$. But we assumed $\varepsilon < 1$. Therefore $\|f-g\| < \varepsilon$.

THEOREM. *Let C be a convex and weakly-compact subset of X. Then C is the closed convex hull of its strongly exposed points and the set S, defined in the introduction, is a dense G_δ subset of the unit sphere $\{f \in X^*; \|f\| = 1\}$ of X^*.*

PROOF. Referring to Lemma 7 of [4], the first assertion is a consequence of Proposition 2. We remark that it also follows from the second part of the theorem.

For $\varepsilon > 0$, let $U(\varepsilon)$ be the set of all $f \in X^*$ such that $\|f\| = 1$ and diam $S(f, \alpha, C) \leqslant \varepsilon$ for some $\alpha > 0$. Then $U(\varepsilon)$ is an open subset of the unit sphere of X^*. Indeed, suppose $f \in U(\varepsilon)$ and diam $S(f, \alpha, C) \leqslant \varepsilon$, then we verify that $S(g, \alpha/3, C) \subset S(f, \alpha, C)$ if $\|g\| = 1$ and $\|f-g\| < \alpha/3M(C)$. It is clear from Proposition 2 that $U(\varepsilon)$ is also dense there.

Since $S = \bigcap_{n=1}^{\infty} U(1/n)$, S is a dense G_δ subset of the unit sphere by the Baire category theorem.

BIBLIOGRAPHY

1. E. Bishop and R. R. Phelps, *The support functionals of a convex set*, Proc. Sympos. Pure Math., vol. 7, Amer. Math. Soc., Providence, R.I., 1963, pp. 27–35. MR **27** #4051.

2. J. Lindenstrauss, *On operators which attain their norm*, Israel J. Math. **1** (1963), 139–148. MR **28** #3308.

3. I. Namioka, *Neighbourhoods of extreme points*, Israel J. Math. **5** (1967), 145–152. MR **36** #4323.

4. R. R. Phelps, *Dentability and extreme points in Banach spaces*, J. Functional Analysis **17** (1974), 78–90. MR **50** #5427.

5. ———, *Lectures on Choquet's theorem*, Van Nostrand, Princeton, N.J., 1966. MR **33** #1690.

6. S. L. Trojanski, *On locally uniformly convex and differentiable norms in certain non-separable Banach spaces*, Studia Math. **37** (1970/71), 173–180. MR **46** #5995.

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