STRONGLY EXPOSED POINTS IN WEAKLY COMPACT CONVEX SETS IN BANACH SPACES

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Abstract. A "purely geometric" proof of the Lindenstrauss-Troyanski result ([2], [6]) on strongly exposed points of weakly compact sets in Banach spaces is given.

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We consider a Banach-space $X$, $\| \cdot \|$. If $\xi \in X$ and if $\epsilon > 0$, then $B(\xi, \epsilon) = \{ x \in X; \| x - \xi \| < \epsilon \}$. The closed convex hull of a set $A \subset X$, is denoted $\overline{c}(A)$. Let $C$ be a convex subset of $X$, then $C^e$ is the set of the extremal points of $C$. If $K$ is a convex, weakly compact, $B$ a convex, closed, bounded set of $X$ and $J$ a closed subinterval of $[0,1]$, $(K, B, J)$ denotes the closed, convex set $\{(1-t)k + tb; k \in K, b \in B, t \in J \}$.

We have to introduce a few geometrical definitions, referring to [4].

Suppose that $C$ is a nonempty, bounded, closed and convex subset of $X$. Let $M(C) = \sup \{ \| x \|; x \in C \}$. If $f \in X^*$ with $\| f \| = 1$, let $M(f, C) = \sup \{ f(x); x \in C \}$, and for each $\alpha > 0$, let $S(f, \alpha, C) = \{ x \in C; f(x) \geq M(f, C) - \alpha \}$. Such a set is called a "slice" of $C$. A point $\xi$ of $C$ is called "strongly exposed" if there exists $f \in X^* (\| f \| = 1)$ such that $\forall \epsilon > 0, \exists \alpha > 0$ with $\xi \in S(f, \alpha, C) \subset B(\xi, \epsilon)$. Let $S$ be the set of all $f \in X^*$ such that $\| f \| = 1$ and $f$ strongly exposes some point of $C$.

Proposition 1. If $K$ is a nonempty, convex, weakly-compact subset of $X$ and if $B$ is convex, closed and bounded with $K \cap B = \emptyset$, then the set $D = \overline{c}(K \cup B) = (K, B, [0,1])$ has the following property: $\forall \epsilon > 0, \exists \xi \in K$ such that $\xi \notin \overline{c}(D \setminus B(\xi, \epsilon))$.

First we observe that it is sufficient to prove the proposition for $X$ separable.

Indeed, suppose $D$ does not have the required property. Then there exists $\epsilon > 0$ with $\forall x \in K: x \in \overline{c}(D \setminus B(x, \epsilon))$ and hence, $\forall x \in K, \exists A(x) \subset D$ with the properties: $x \in \overline{c}(A(x))$, $A(x) \cap B(x, \epsilon) = \emptyset$ and $A(x)$ is countable. By induction we construct a sequence $(K_n, B_n)_n$ where $K_n$ is countable in $K$ and $B_n$ is countable in $B$.

Let $(K_0, B_0) = (\{ x \}, \emptyset)$, where $x$ is some element of $K$. Suppose we already found $(K_n, B_n)$.

Consider $\forall y \in \bigcup_{x \in K_n} A(x)$ an element $k_y$ in $K$ and $b_y$ in $B$, with $y \in \overline{c}(\{ k_y, b_y \})$.

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Then $K_{n+1}$ and $B_{n+1}$ are still countable, which completes the construction. Consider then $K' = \bigcap_{n} K_n$, $B' = \bigcup_{n} B_n$, $D' = \overline{c}(K' \cup B')$ and the closed linear span $X_0$ of $D'$.

Since $\forall n: \overline{c}(K_n) \subset K_{n+1}$, $K'$ is convex and closed. The same holds for $B'$. $X_0$ is a separable Banach-space wherein $K'$ is nonempty, convex, weakly-compact and $B'$ convex, closed and bounded. We show that $D' \subset X_0$ does not have the property mentioned in the proposition. Choose $x' \in K'$ and take $0 < \delta < \varepsilon/2$. There exist $n \in \mathbb{N}$ and $x \in K_n$ with $\|x - x'\| < \delta$. We obtain:

$$x \in \overline{c}(A(x)) \subset \overline{c} \left( \bigcup_{y \in A(x)} \overline{c}((k_y, b_y)) \backslash B(x, \varepsilon) \right) \subset \overline{c}(D' \backslash B(x, \varepsilon))$$

Since this holds, $\forall \delta (0 < \delta < \varepsilon/2)$, we have $x' \in \overline{c}(D' \backslash B(x', \varepsilon/2))$. $D'$ does not have the property and we can restrict ourselves to the case of a separable Banach-space.

**Lemma.** $D = \overline{c}(D' \cup B)$.

**Proof.** Clearly $E = \overline{c}(D' \cup B) \subset D$. For the reverse inclusion it is sufficient to prove that $K \subset E$. Suppose that $x_0 \in K \setminus E$. Then $\exists u \in X^*$ such that $u(x_0) > \sup \{u(x); x \in D' \cup B\} \geq \sup \{u(x); x \in B\}$. Let $\alpha = \sup \{u(x); x \in K\}$, and let $y \in D$. Then $y = \lambda x_1 + (1 - \lambda) x_2$ with $x_1 \in K$, $x_2 \in B$, $\lambda \in [0,1]$. It follows that

$$u(y) = \lambda u(x_1) + (1 - \lambda) u(x_2) \leq \lambda u(x_1) + (1 - \lambda) u(x_0)$$

$$= u(\lambda x_1 + (1 - \lambda) x_0) \leq \alpha.$$ 

This shows that $\sup \{u(y); y \in D\} = \alpha$, and, since $u(x_2) \leq u(x_0) \leq \alpha$, $u(y) = \alpha$ implies that $\lambda = 1$ or $y \in K$. Let $D_1 = \{ y \in D; u(y) = \alpha \}$. Then $D_1$ is a supporting set of $D$, and $D_1 \subset K$. Since $D_1$ is weakly compact, it contains an extreme point $z$. Then $z \in D'$, but since $u(z) = \alpha$, $z \not \in (D' \cup B)$. This contradiction establishes that $K \subset E$ or $D = E$.

**Proof of Proposition 1.** The proof is a modification of a proof in Namioka [3]. We assume that $X$ is separable. By the lemma, $A = D' \cup B \neq \emptyset$. Since $A \subset K$ the weak closure $\overline{A}$ of $A$ is a Baire space relative to the weak topology. Since $X$ is separable, there is a sequence $(x_n)$ in $X$ such that
Note that each $B(x_n, \varepsilon/4)$ is weakly closed. Hence there is a weakly open set $N$ in $X$ such that $N \cap \overline{A} \neq \emptyset$ and $N \cap \overline{A} \subset B(x_n, \varepsilon/4)$ for some $n$. Then clearly $\text{diam}(N \cap \overline{A}) \leq \varepsilon/2$. Let

$$D_1 = \overline{c}(N \cap \overline{A}) \quad \text{and} \quad D_2 = \overline{c}((\overline{A} \setminus N) \cup B).$$

Then we observe that $D_1$ is weakly compact and $\text{diam} D_1 \leq \varepsilon/2$.

Since $A$ is weakly dense in $A$, $N \cap A \neq \emptyset$. Fix $\xi \in N \cap A \subset K$. Then $\xi \notin D_2$. For $\xi \in D_2$, then $\xi \in \overline{c((K \setminus N) \cup B)} \subset \overline{c(K \setminus N)} \subset B(0, 1]$. Since $\xi \in D^e$ and $\xi \notin B$, we have $\xi \in \overline{c(K \setminus N)}$. It then follows from the Krein-Milman theorem that $\xi \in K \setminus N$, because $K \setminus N$ is weakly compact. This contradicts $\xi \in N \cap A$.

Because $D = \overline{c}(D^e \cup B)$, $D = \overline{c}(D_1 \cup D_2) = (D_1, D_2, [0, 1])$. Let $\mathcal{C} = (D_1, D_2, [\varepsilon/5d, 1])$, where $d = \text{diam} D$. Then $C$ is a closed convex subset of $D$.

If $\xi \in C$, then $\xi = (1 - \lambda)x_1 + \lambda x_2$ where $x_1 \in D_1$, $x_2 \in D_2$ and $\lambda \in [\varepsilon/5d, 1]$. Since $\xi$ is extreme, this implies that $\xi \in D_2$, which is impossible. Hence $\xi \notin C$. Let $y_1, y_2 \in D \setminus C$. Then $y_i = (1 - \lambda_i)x_i + \lambda_i x_i'$, where $x_i \in D_1$, $x_i' \in D_2$ and $\lambda_i \in [0, \varepsilon/5d]$ ($i = 1, 2$). We then have:

$$\|y_1 - y_2\| \leq \|x_1 - x_2\| + \lambda_1\|x_1' - x_2\| + \lambda_2\|x_1' - x_2\| \leq \varepsilon/2 + \varepsilon d/5d + \varepsilon d/5d = 9\varepsilon/10.$$ 

Since $\xi \in D \setminus C$ and $\text{diam} (D \setminus C) < \varepsilon$, it follows that $D \setminus C \subset B(\xi, \varepsilon)$. Therefore $D \setminus B(\xi, \varepsilon) \subset C$ and $\overline{c}(D \setminus B(\xi, \varepsilon)) \subset C$. Thus $\xi \notin \overline{c}(D \setminus B(\xi, \varepsilon))$ and the proof is complete.

Remark. The fact that $X, \|\| \|$ is complete is not used and the assertion is still true when $X$ is only a normed space.

Proposition 2. Let $C$ be a convex and weakly-compact subset of $X$. If $S(f, \alpha, C)$ is a slice of $C$, then $\forall \varepsilon > 0: \exists g \in X^*$, $\exists \beta > 0$ such that $S(g, \beta, C)$ is a slice of $C$ with diameter $\leq \varepsilon$, $S(g, \beta, C) \subset S(f, \alpha, C)$ and $\|f - g\| \leq \varepsilon$.

Proof. The proof is a modification of a proof in Phelps [4]. By translation we may assume that $0 \in H = \{x \in X; f(x) = M(f, C) - \alpha\}$. Hence $H = f^{-1}(0)$ and $\alpha = M(f, C) > 0$. We may also assume that $\varepsilon < \min(1, \alpha)$. Choose $\lambda$ so that $\lambda > 2M(C)/\varepsilon$, and let $K = S(f, \alpha/2, C)$ and $B = H \cap \overline{B}(0, \lambda)$. Since $K \cap B = \emptyset$ and $K \neq \emptyset$, we may apply Proposition 1 to $K$ and $B$.

Let $D = \overline{c}(K \cup B)$. Then $\exists \xi \in K$ such that $\xi \notin \overline{c}(D \setminus B(\xi, \varepsilon/2))$. If $x \in B$, then $\|x - \xi\| > |f(x - \xi)| > \alpha/2 > \varepsilon/2$. Therefore $B \subset \overline{c}(D \setminus B(\xi, \varepsilon/2))$. By the separation theorem, $\exists g \in X^*$ such that $\|g\| = 1$ and $g(\xi) > \sup\{g(x); x \in D \setminus B(\xi, \varepsilon/2)\} > 0$. Since $M(g, C) > g(\xi)$, we may write $\sup\{g(x); x \in D \setminus B(\xi, \varepsilon/2)\} = M(g, C) - 2\beta$ with $\beta > 0$.

Then $S(g, \beta, C) \subset B(\xi, \varepsilon/2)$, and hence $\text{diam} S(g, \beta, C) \leq \varepsilon$. Suppose that $f(x) = 0$ and $\|x\| \leq 1$. Then $\lambda x \in B$, and hence $\lambda g(x) < g(\xi)$ or $g(x) < \lambda^{-1} g(\xi)$. By Lemma 2 of [1], this implies that either $\|f - g\| \leq 2\lambda^{-1} g(\xi)$
\[ 2\lambda^{-1} M(C) < \varepsilon \text{ or } \|f + g\| \leq 2\lambda^{-1} g(\xi). \] If the second possibility occurs, then \[ g(\xi) \leq (f + g)(\xi) \leq \|f + g\| \leq 2\lambda^{-1} g(\xi) M(C) < g(\xi) \varepsilon, \] which implies that \(1 < \varepsilon\). But we assumed \(\varepsilon < 1\). Therefore \(\|f - g\| < \varepsilon\).

**Theorem.** Let \( C \) be a convex and weakly-compact subset of \( X \). Then \( C \) is the closed convex hull of its strongly exposed points and the set \( S \), defined in the introduction, is a dense \( G_\delta \) subset of the unit sphere \( \{ f \in X^* ; \|f\| = 1 \} \) of \( X^* \).

**Proof.** Referring to Lemma 7 of [4], the first assertion is a consequence of Proposition 2. We remark that it also follows from the second part of the theorem.

For \( \varepsilon > 0 \), let \( U(\varepsilon) \) be the set of all \( f \in X^* \) such that \( \|f\| = 1 \) and \( \text{diam } S(f, \alpha, C) \leq \varepsilon \) for some \( \alpha > 0 \). Then \( U(\varepsilon) \) is an open subset of the unit sphere of \( X^* \). Indeed, suppose \( f \in U(\varepsilon) \) and \( \text{diam } S(f, \alpha, C) \leq \varepsilon \), then we verify that \( S(g, \alpha/3, C) \subseteq S(f, \alpha, C) \) if \( \|g\| = 1 \) and \( \|f - g\| < \alpha/3 M(C) \). It is clear from Proposition 2 that \( U(\varepsilon) \) is also dense there.

Since \( S = \bigcap_{n=1}^{\infty} U(1/n) \), \( S \) is a dense \( G_\delta \) subset of the unit sphere by the Baire category theorem.

**Bibliography**


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