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A FORMULA FOR STIEFEL-WHITNEY HOMOLOGY CLASSES

RICHARD Z. GOLDSTEIN[1] AND EDWARD C. TURNER

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The purpose of this paper is to define for mod 2 Euler spaces a formula which enables one to compute the *Stiefel-Whitney homology classes* in the original triangulation without passing to the first barycentric subdivision. The formula has a somewhat tenuous connection to the Steenrod reduced squares. In the case when we are dealing with a smooth triangulation, the Wu formulae [7] and the Whitney theorem [4] establish such a connection. The authors would like to thank S. Halperin and D. Toledo for a copy of their preprint [5]; the use of their map ϕ (see §2) simplifies an earlier proof of the main theorem. The homology theory used is that based on infinite chains.

1. **Statement of the theorem.** Let K be a finite-dimensional, locally finite simplicial complex. K is said to be a mod 2 Euler space if the link of every simplex in K has even Euler characteristic [9]. The pth Stiefel-Whitney class of K, denoted $\omega_p(K)$, is the p-dimensional mod 2 homology class which has a representative, the p-dimensional chain consisting of all p-simplexes in the first barycentric subdivision of K–this chain is a cycle for each p iff K is a mod 2 Euler space.

From now on we assume that K is given an ordering of its vertices and any representation of a simplex in K is written with its vertices in *increasing order*. We now recall a definition due to Steenrod [8]. Let s be a p-simplex in K, say $s = \langle v_0, v_1, \ldots, v_p \rangle$. Let t be another simplex which has s as a face; i.e., $s \subset t$ (s may be equal to t). Let

B_{-1} = set of vertices of t less than v_0,
B_0 = set of vertices of t strictly between v_0 and v_1,
B_m = set of vertices of t strictly between v_m and v_{m+1},
B_p = set of vertices of t greater than v_p.

We say that s is *regular* in t, if $\sharp\,(B_m) = 0$ for every odd m. Let $\partial_p(t)$ denote the mod 2 chain which consists of all p-dimensional simplexes s in t so that s

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is regular in t. The following is the main result in this paper.

THEOREM. *$\sum_{\dim t \geqslant p} \partial_p(t)$ is a chain which represents $\omega_p(K)$.*

2. **Proof of the main theorem.** We define a simplicial map $\phi\colon K' \to K$ in the following way. Pick a simplex s in K and let $l(s)$ be the smallest vertex in s; map the barycenter of s to $l(s)$. This defines a simplicial map $\phi\colon K' \to K$ which induces the identity in homology. To establish the theorem it suffices to show that the number of p-simplexes in K' which map *onto* a given p-simplex s in K is congruent mod 2 to the number of simplexes t, so that s is regular in t. However a p-simplex s' in K' maps onto s only if s is a face of the carrier in K of s'; hence the theorem is a consequence of the following

LEMMA. *Let $\phi\colon K' \to K$ be defined as above and let s be a face of t; then the number of p-simplexes whose carrier is t and which map onto s is*
(i) *odd if s is regular in t,*
(ii) *even if s is not regular in t.*

PROOF. If B_{-1} is not empty then s is not regular in t and no simplex whose carrier is t maps onto s. Hence we assume that B_{-1} is empty. Now a p-simplex whose carrier is t and whose image under ϕ is s must look like the following:

$$(\{v_0, B_0, v_1, B_1, \ldots, v_p, B_p\}, \{v_1, B'_1, v_2, \ldots, v_p, B'_p\}, \ldots, \{v_p, B_p^{(p)}\})$$

where

$$\begin{gathered} B'_1 \subset B_1, \\ B''_2 \subset B'_2 \subset B_2, \\ \vdots \\ B_p^{(p)} \subset \cdots \subset B'_p \subset B_p. \end{gathered}$$

Denote by $c(B_j)$, the number of ways of choosing j nondecreasing subsets of B_j; hence the number of p-simplexes whose image is s is $c(B_1) \cdot c(B_2) \cdot \cdots \cdot c(B_p)$.

PROPOSITION. (1) $c(B_j) = 1$ *if* $B_j = \varnothing$,
(2) $c(B_j) \equiv 0 \pmod 2$ *if* $B_j \neq \varnothing$ *and j is odd,*
(3) $c(B_j) \equiv 1 \pmod 2$ *if* $B_j \neq \varnothing$ *and j is even.*

PROOF. Part 1 follows from the fact that $B_j^{(k)}$ is always the null set.
Now when $m \geqslant 0$ we have that

$$\begin{aligned} \sum_{a=0}^{m} \sum_{b=0}^{a} \binom{m}{a}\binom{a}{b} &= \sum_{a=0}^{m} \binom{m}{a} \sum_{b=0}^{a} \binom{a}{b} \\ &\equiv \binom{m}{0} \pmod 2 \equiv 1 \pmod 2. \end{aligned}$$

Now in general

$$c(B_j) = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n_1} \cdots \sum_{n_j=0}^{n_{j-1}} \binom{n}{n_1}\binom{n_1}{n_2} \cdots \binom{n_{j-1}}{n_j} \quad \text{where } \sharp\,(B_j) = n_j.$$

Therefore, mod 2 we have that

$$c(B_j) \equiv 1 \quad \text{when } j \text{ is even.}$$

$$c(B_j) = \sum_{n_1=0}^{n} \binom{n}{n_1} \equiv 0 \quad \text{when } j \text{ is odd since } n > 0.$$

Now back to the proof of the lemma. When s is regular in t, $B_j = \varnothing$ for j odd; hence

$$c(B_1) \cdot c(B_2) \cdot \cdots \cdot c(B_p) \equiv 1 \cdot 1 \cdot \cdots \cdot 1 \equiv 1 \mod 2.$$

When s is not regular in t, some $B_j \neq \varnothing$ for j odd; hence $c(B_j) \equiv 0$ and the lemma is proven.

3. **Some remarks on the formula.** The above proof works verbatim in the case that the vertices are partially ordered in such a way that those of any simplex are linearly ordered, as in the barycentric subdivision ordered by dimension. It is not hard to see that in K', each p-simplex is a regular face of an odd number of simplexes, so that

$$\sum_{\dim(t) \geqslant p} \partial_p(t) = \sum p\text{-simplexes of } K'.$$

It follows that the p-skeleta of repeated barycentric subdivisions are homologous. (E. Akin has shown that $\omega_p(K)$ is a PL invariant [1].)

When $p = 0$ it is tempting to call the coefficient of a vertex v in

$$\sum_{\dim t \geqslant 0} \partial_0(t)$$

the local Euler number. The reason is two-fold: first the number of vertices whose coefficient is 1 is congruent mod 2 to $\chi(K)$. Secondly when K is an immersed surface in R^3 and the ordering on the vertices is induced by projection on the z-axis, then the vertices with coefficient 1 are precisely the critical points of the function defined on K by this projection. Thus we get that $\chi(K) \equiv$ number critical pts (mod 2).

When K is n-dimensional, then the ordering on the vertices induces an orientation on each simplex in K. Now when s is an $(n-1)$-simplex and a *proper* face of t, we have that s is regular in t if and only if the orientation on t induces the opposite orientation on s. Thus $\omega_{n-1}(K)$ has as a representative the sum of those $(n-1)$-simplexes whose orientation disagrees with an even number of n-simplexes of which it is a face (a simplex is always regular in itself). This last fact is exploited in [3].

In fact our formula can be derived from the work of Banchoff [2] and McCrory [6]. Given an ordering of the vertices of K define a vertex map into R^m by sending the vertex j into $(j, j^2, \ldots, j^m)$. This defines a full map and some combinatorics applied to the description of the S.W. classes as given in [6] gives our formula.

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MATHEMATISCHES INSTITUT, UNIVERSITÄT WÜRZBURG, WÜRZBURG, FEDERAL REPUBLIC OF GERMANY

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK, ALBANY, NEW YORK 12222