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ON THE NUMBER OF SEPARABLE LOCALLY CONVEX SPACES

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ABSTRACT. The number of distinct separable locally convex spaces is shown to be 2^{2^c}. The number of distinct separable and complete, or metrizable, or normed locally convex spaces is shown to be 2^c. There is no separable locally convex space that is quotient-universal for the class of separable locally convex spaces.

1. **Introduction.** Throughout this paper we consider only Hausdorff locally convex spaces over the fixed field K of either real or complex numbers. Two locally convex spaces E and F are called isomorphic, denoted by $E \cong F$, if there exists an isomorphism (i.e., a linear homeomorphism) mapping E onto F.

Let $\mathfrak{L}$ be a class of locally convex spaces. A space E belonging to $\mathfrak{L}$ is *subspace universal* (respectively, *quotient universal*) in $\mathfrak{L}$ if for each L in $\mathfrak{L}$ there exists a subspace F of E such that $L \cong F$ (respectively, $L \cong E/F$). In order to shorten our formulations, we denote by $\mathfrak{S}$ the class of all separable locally convex spaces, and by $\mathcal{C}$, $\mathfrak{M}$, $\mathfrak{F}$, $\mathfrak{N}$, and $\mathfrak{B}$ the classes of all complete, metrizable, metrizable complete (Fréchet), normed, and normed complete (Banach) spaces in $\mathfrak{S}$, respectively.

First, let us recall a few known facts. By the classical results of Banach and Mazur [1, p. 185], [2], the spaces $C = C[0, 1]$ and l_1 are, respectively, subspace universal in $\mathfrak{N}$ and quotient universal in $\mathfrak{B}$. Mazur and Orlicz [9, p. 14] proved that the Fréchet space $C(R)$ of all continuous functions $f\colon R \to K$, endowed with the topology of uniform convergence on compact sets, or, more abstractly, the product $C^{\aleph_0}$ of countably many copies of C, is subspace universal in $\mathfrak{M}$. More recently, it was observed in [7] that the product C^c of c (continuum) copies of C serves as a subspace universal space in $\mathfrak{S}$.

It is unknown whether the classes $\mathfrak{S}$, $\mathcal{C}$, $\mathfrak{M}$, $\mathfrak{N}$, or $\mathfrak{F}$ possess a quotient universal space (cf. [11, p. 47]). We show that the answer is negative (Corollary 1) in the cases of $\mathfrak{S}$, $\mathfrak{M}$, and $\mathfrak{N}$. This follows as an easy consequence of the main result:

THEOREM. $\mathfrak{S}$ *contains a subset* $\mathfrak{I}$ *consisting of* 2^{2^c} *mutually nonisomorphic spaces such that each member of* $\mathfrak{S}$ *is isomorphic to a member of* $\mathfrak{I}$. *The same assertion holds for* $\mathcal{C}$, $\mathfrak{M}$, *and* $\mathfrak{N}$ *with the cardinality of* $\mathfrak{I}$ *changed to* 2^c.

In addition to yielding the number of "distinct" separable locally convex spaces, the method of proof of the Theorem also shows that every subspace of

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finite codimension of a separable locally convex space is separable (Corollary 3). An example shows this is not true for subspaces of countable codimension.

2. The main results.

PROOF OF THEOREM. Let $E = (E, t)$ be a locally convex space, $E' = (E, t)'$ its topological dual, and E^* its algebraic dual. For each f in E^*, let t_f denote the weakest (locally convex) linear topology on E such that $t \leqslant t_f$ and f is continuous under t_f. Thus, if $E_f' = \operatorname{sp}(E' \cup \{f\})$, then $t_f = \sup\{t, \sigma(E, E_f')\}$, so that $E_f' = (E, t_f)'$.

By definition $t \leqslant t_f$, and the equality holds iff $f \in E'$. Note also that on $f^{-1}(0)$ the topologies t and t_f coincide, and since $f^{-1}(0)$ is a closed hyperplane in (E, t_f), it follows that

$$(1) \qquad (E, t_f) \cong f^{-1}(0) \oplus K,$$

where $f^{-1}(0)$ is endowed with the topology induced by t.

If $f, g \in E^*$, then $g \in E_f'$ iff $E_g' \subset E_f'$ and $E_g' \subset E_f'$ iff $t_g \leqslant t_f$. Since $E_f' = E'$ or E' is of codimension one in E_f', $E_g' \subset E_f'$ holds iff $E_g' = E'$ or $E_g' = E_f'$. Hence $t_g \leqslant t_f$ iff $t_g = t$ or $t_g = t_f$. Therefore, if $g \notin E'$, then $t_g \leqslant t_f$ iff $g \in E_f' \backslash E'$. Writing $f \sim g$ when $t_f = t_g$, we define an equivalence relation on E^*; let $[f]_\sim = \{g \in E^* : f \sim g\}$. Then $[f]_\sim = E'$ for each $f \in E'$, $[f]_\sim = E_f' \backslash E'$ for each $f \in E^* \backslash E'$, and $\operatorname{card}[f]_\sim = \operatorname{card} E'$ for each $f \in E^*$.

From now on let us assume that (E, t) is separable. Then, for each $f \in E^*$, the space (E, t_f) is separable, too. In fact, let D be a countable dense subset of E, and suppose D is not dense in (E, t_f). Let L be the t_f-closed linear subspace of E spanned by D. By assumption, $L \neq E$. Hence there exists a nonzero g in E_f' such that $g(L) = \{0\}$. Since $g \notin E'$, we have $t_g = t_f$, and so by (1)

$$(E, t_f) = (E, t_g) \cong g^{-1}(0) \oplus K.$$

But $g^{-1}(0)$, in the topology induced by t, is evidently separable, hence (E, t_f) is separable as well.

Assume $\dim E = m$, where $m = c$ or $m = 2^c$. Then $\operatorname{card} E^* = \dim E^* = 2^m$ and $\operatorname{card} E' = c$ because E is separable. From each equivalence class $[\ ]_\sim \neq E'$ considered above choose an element f, and let F denote the set thus obtained. (Instead, we may take any Hamel basis B of E^*/E' and then define F to be a set of representatives chosen out of each member of B.) Evidently $\operatorname{card} F = 2^m$, because $\operatorname{card}[f]_\sim = c$ for each $f \in E^*$. Moreover, F has the property

$$(2) \qquad f, g \in F \text{ and } f \neq g \text{ imply } t_f \text{ and } t_g \text{ are totally incomparable;}$$

that is, neither $t_f \leqslant t_g$ nor $t_g \leqslant t_f$ holds.

Now, given an f in F, we claim that (E, t_f) can be isomorphic to (E, t_g) for at most m members g of F. In order to see this, let D be a countable dense subset of (E, t_f). If, for some $g \in F$, A is an isomorphism of (E, t_f) onto (E, t_g), then A is completely determined by the values it assumes on D. Hence there are at most $m^{\aleph_0} = m$ distinct mappings A of E into E such that for some $g \in F$ (depending on A) A establishes the isomorphism of (E, t_f) and (E, t_g). It is not excluded, however, that different g's may correspond to the same A.

So suppose $g, h \in F$, $g \neq h$, and A is an isomorphism of (E, t_f) onto (E, t_g) and onto (E, t_h). Then the identity AA^{-1} is an isomorphism of (E, t_g) onto (E, t_h), and so $t_g = t_h$, which contradicts property (2) of F.

Now, for any $f, g \in F$, let $f \approx g$ mean that (E, t_f) and (E, t_g) are isomorphic. By what was just shown, each equivalence class $[f]_{\approx}$ contains at most m members of F, and since card $F = 2^m$, there are precisely 2^m distinct $\approx$-equivalence classes. Thus there exists a subset G of F with card $G = 2^m$ such that the separable locally convex spaces (E, t_g), $g \in G$, are mutually nonisomorphic. These spaces are metrizable or normed if such is the space E.

Now specifying E to be C^c, we have dim $C^c = 2^c$. The universality of E guarantees the existence of a maximal set $\mathcal{J}$ of mutually nonisomorphic linear subspaces of E such that each member of $\mathcal{S}$ is isomorphic to a member of $\mathcal{J}$. Clearly, we must have card $\mathcal{J} \leqslant 2^{2^c}$. On the other hand, each space (E, t_g), $g \in G$, is isomorphic to a member of $\mathcal{J}$ so that card $\mathcal{J} \geqslant 2^{2^c}$.

In the case of $\mathcal{M}$ and $\mathcal{N}$, let E be $C^{\aleph_0}$ and C, respectively, and note that dim $C^{\aleph_0} =$ dim $C = c$. In either case, let $\mathcal{J}$ be defined in a similar manner as before. Then card $\mathcal{J} \leqslant 2^c$ and a similar argument as before shows card $\mathcal{J} \geqslant 2^c$.

In the case of $\mathcal{C}$, let E be C^c and choose $\mathcal{J}$ to be a maximal set of mutually nonisomorphic closed linear subspaces of E. Since each closed linear subspace of E is an intersection of closed hyperplanes and card $E' = c$, we have card $\mathcal{J} \leqslant 2^c$. On the other hand, for each nonempty subset P of $[1, \infty)$, let l_P denote the product $\prod_{p \in P} l_p$. Then l_P is complete and separable by the Hewitt-Marczewski-Pondiczery theorem (cf. [5, p. 77], [10]). If P, $Q \subset [1, \infty)$ and $P \neq Q$, then l_P and l_Q are not isomorphic. Indeed, suppose $P \not\subset Q$ and let $p \in P \setminus Q$. If A is an isomorphism of l_P onto l_Q, then its restriction to the factor space l_p is an embedding of l_p into l_Q. It follows from Proposition 3 of [3] that there exist $q_1, \ldots, q_n \in Q$ such that $\prod_{i=1}^n l_{q_i}$ contains an isomorphic copy X of l_p. As l_p and l_{q_i} are well known to have no infinite-dimensional isomorphic subspaces, each coordinate projection of $\prod_{i=1}^n l_{q_i}$ onto X is strictly singular. However, the sum of strictly singular maps is strictly singular (cf. [6, p. 86]) so that the identity mapping on X is strictly singular. The contradiction shows that l_P and l_Q are not isomorphic. Since $\mathcal{J}$ must contain an isomorphic copy of l_P for $\varnothing \neq P \subset [1, \infty)$, card $\mathcal{J} \geqslant 2^c$.

Remark. It is well known that a set $\mathcal{J}$ satisfying the statement of the Theorem can be chosen for the classes $\mathcal{F}$ and $\mathcal{B}$ with card $\mathcal{J} = c$.

Corollary 1. *There is no quotient-universal space in the classes* $\mathcal{N}$, $\mathcal{M}$, *and* $\mathcal{S}$.

Proof. Consider, for instance, the case of $\mathcal{S}$. Suppose that a space $E \in \mathcal{S}$ is quotient-universal in $\mathcal{S}$. Then E admits at most 2^c distinct (up to isomorphism) Hausdorff quotients, because card $E' = c$ and each closed linear subspace of E is an intersection of a family of closed hyperplanes $f^{-1}(0)$, $f \in E'$. But there are 2^{2^c} mutually nonisomorphic separable locally convex spaces, yielding a contradiction. The proof in the cases of $\mathcal{M}$ and $\mathcal{N}$ is similar.

By a result of Dieudonné [4], every finite-codimensional subspace of a barrelled space is barrelled (and hence a Mackey space). The subspace-universal spaces E used in the proof of the Theorem are barrelled as products of barrelled spaces, and the spaces (E, t_f) are barrelled since they are isomorphic to products of one-codimensional subspaces of E with K. Hence we have

COROLLARY 2. *There are precisely* 2^{2^c}, 2^c *and* 2^c *distinct barrelled (resp. Mackey) spaces in* $\mathbb{S}$, $\mathfrak{M}$, *and* $\mathfrak{N}$, *respectively.*

Our next result is also an easy consequence of the proof of the Theorem (cf. [14]).

COROLLARY 3. *Every subspace of finite codimension of a separable locally convex space is separable.*

REMARK. In view of some recent results showing that certain properties of locally convex spaces are inherited by subspaces of countable codimension (see e.g. [12] and [13] for the property of being barrelled), one might ask whether, in Corollary 3, "finite" can be replaced by "countable". We give a counterexample showing that this need not be the case. Note that there exists a separable locally convex space E with a closed nonseparable subspace F. An example of such a space was given in [8]. Let S be a countable subset of E whose closed linear span is all of E. Then $G = \operatorname{sp}(F \cup S)$, in the relative topology, is a separable locally convex space whose countable-codimensional closed subspace F is not separable.

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