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OPERATORS SATISFYING CERTAIN GROWTH CONDITIONS. II

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ABSTRACT. It is proved that the condition $w_\rho[(T - zI)^{-1}] = 1/d(z, \sigma(T))$, $w_\rho(\cdot)$ being the operator radius of Holbrook, implies the existence of certain eigenvalues and normal eigenvalues for a Hilbert space operator T. This extends known results based on a norm condition ($\rho = 1$) and allows a similar extension of various consequences of these results.

Let C_ρ ($\rho > 0$) denote the set of all operators on a Hilbert space H with unitary ρ-dilation in the sense of [1]. According to Holbrook [2] an operator radius of T is defined by

$$w_\rho(T) = \inf\left\{\alpha: \alpha > 0 \text{ and } \alpha^{-1}T \in C_\rho\right\}.$$

In particular, $w_1(T) = \|T\|$ and $w_2(T) = |W(T)|$, the numerical radius of T. An operator T is called ρ-oid if $w_\rho(T) = r(T)$, where $r(T)$ is the spectral radius of T.

T is said to satisfy condition $G_k(n)$ with respect to X if

$$\|(T - zI)^{-1}\| \leqslant k/d(z, X)^n \qquad \text{for all } z \notin X,$$

where X is a closed set containing the spectrum $\sigma(T)$ of T, $d(z, X)$ is the distance from z to X, and n is a positive integer. The condition $G_1(1)$ with respect to $\sigma(T)$ is the same as the usual condition (G_1) of [3].

An operator T is said to be of class M_ρ ($\rho \geqslant 1$) if

$$w_\rho\left[(T - zI)^{-1}\right] = 1/d(z, \sigma(T));$$

equivalently, $(T - zI)^{-1}$ is ρ-oid for all $z \notin \sigma(T)$.

Clearly, the class M_1 consists of all operators which satisfy condition (G_1) and every operator in M_ρ satisfies the condition $G_\rho(1)$ with respect to $\sigma(T)$. The classes M_ρ ($\rho \geqslant 1$) are contained in the class of all convexoid operators [5] and form a nondecreasing family with respect to ρ. For further properties of operators in M_ρ we refer to [5] and [6].

Stampfli has proven the following results, respectively, in [3] and [4].

(1) If T satisfies condition (G_1) then T is isoloid, that is, every isolated point of $\sigma(T)$ is an eigenvalue.

(2) If $\sigma(T)$ lies in a C^1-Jordan curve Γ and T satisfies condition $G_1(1)$ outside and condition $G_k(n)$ inside Γ with respect to Γ, then T is isoloid.

In this note we prove that the same conclusion is obtained for an operator T of class M_ρ without imposing any condition on its spectrum. This leads to

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several corollaries extending various known results based on the norm condition $G_1(1)$.

In fact, we can prove the following stronger result.

THEOREM 1. *If for some $k < \infty$ and $\rho \geqslant 1$,*

$$w_\rho\left[(T - zI)^{-1}\right] \leqslant k/d(z, \sigma(T)), \qquad z \not\in \sigma(T),$$

then every isolated point z_0 of $\sigma(T)$ is an eigenvalue.

PROOF. We may assume $z_0 = 0$. Let $P = (2\pi i)^{-1}\int_{C_R}(T - zI)^{-1}\,dz$, C_R being a circle of radius R, be the spectral projection corresponding to the isolated point $z_0 = 0$ of $\sigma(T)$. Then $x \in P(H)$ implies

$$Tx = \frac{1}{2\pi i}\int_{C_R} z(T - zI)^{-1}x\,dz,$$

so that

$$\|Tx\| \leqslant \|x\| R^2 \max_{z \in C_R} \left\|(T - zI)^{-1}\right\|.$$

But $\|S\| \leqslant \rho w_\rho(S)$ for every operator S and so

$$\begin{aligned}\|Tx\| &\leqslant \|x\|\rho R^2 \max_{z \in C_R} w_\rho\left[(T - zI)^{-1}\right]\\ &= \|x\|\rho K R^2 \max_{z \in C_R} (d(z, \sigma(T)))^{-1} = \|x\|\rho K R\end{aligned}$$

for small R. Letting $R \to 0$ we see that $Tx = 0$. This completes the proof.

The following theorem, which will be used strongly in our corollaries, has been proved by Patel [5], and for the sake of completeness we include its proof here. Recall that z is a normal approximate eigenvalue of T if there exists a sequence $\{x_n\}$ of unit vectors such that $(T - z)x_n \to 0$ and $(T^* - \bar{z})x_n \to 0$.

THEOREM 2. *Let $T \in M_\rho$. If z_0 is a semibare point of $\sigma(T)$, then it is a normal approximate eigenvalue of T.*

PROOF. We may assume $z_0 = 0$. Let $u \neq 0$ be such that $\{z\colon |z - u| \leqslant |u|\} \cap \sigma(T) = \{0\}$. Then $d(u, \sigma(T)) = |u|$ and $w_\rho[(T - uI)^{-1}] \leqslant |u|^{-1}$. Set $S = -u(T - uI)^{-1}$. Since 0 is in the boundary of $\sigma(T)$, 0 is an approximate eigenvalue of T. If $\{x_n\}$ is a sequence of unit vectors such that $Tx_n \to 0$ then $Sx_n - x_n \to 0$. Since $w_\rho(S) \leqslant 1$, $S \in C_\rho$. It follows from [1, Theorem I.11.1] that $\operatorname{Re} V \geqslant 0$ where

$$V = (\rho - 2)(I - S^*)(I - S) + 2(I - S).$$

Therefore

$$|((\operatorname{Re} V)x, y)|^2 \leqslant ((\operatorname{Re} V)x, x)\,((\operatorname{Re} V)y, y)$$

for all x, y in H.

Taking $x = x_n$, $y = (\operatorname{Re} V)x_n$ and noting that $Vx_n \to 0$, the above inequality yields $(\operatorname{Re} V)x_n \to 0$ and so $V^*x_n \to 0$. Using $(I - S)x_n \to 0$ it follows that $(I - S^*)x_n \to 0$ and hence $T^*x_n \to 0$.

COROLLARY 1. *If $T \in M_\rho$ and $\sigma(T)$ is connected, then* $\operatorname{Re}\sigma(T) = \sigma(\operatorname{Re} T)$.

PROOF. By Theorem 2, every semibare point of $\sigma(T)$ is a normal approximate eigenvalue of T and hence the proof is on the same lines as in Theorem 4 of Berberian [9].

COROLLARY 2. *If $T \in M_\rho$ and $\sigma(T)$ is finite, then T is normal.*

PROOF. Since every point z_i of $\sigma(T)$ is isolated, by Theorems 1 and 2, z_i is a reducing eigenvalue of T. Let E_i denote the spectral projection corresponding to z_i. Then $T = \sum_{i=1}^{n} z_i E_i$ as in [3] and hence T is normal.

COROLLARY 3. *If T is reduction M_ρ, that is, every direct summand of T is in M_ρ, and every point of $\sigma(T)$ is a semibare point, then Weyl's theorem holds for T.*

PROOF. It follows from Theorems 1 and 2 that T is reduction isoloid and each eigenspace of T is reducing. Hence the result is a consequence of Theorem 5.3 of [7].

COROLLARY 4. *If T is reduction M_ρ and $\sigma(T)$ is countable, then T is a diagonal operator.*

PROOF. By virtue of Theorems 1 and 2, the proof is just an imitation of that of Theorem 1 in [8].

Lastly, we remark that all the corollaries of [8] remain valid if the condition (α') is replaced by condition 'reduction M_ρ' for T.

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