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LOCAL MAXIMA OF THE SAMPLE FUNCTIONS OF THE TWO-PARAMETER WIENER PROCESS

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ABSTRACT. It is shown that for almost all sample functions of the two-parameter Wiener process, the set of local maxima is dense in R_+^2.

Let $\{X(t): t \in R_+^2\}$ be the two-parameter Wiener process, that is a real valued Gaussian process with zero means and covariance

$$\min(s_1, s_2)\min(t_1, t_2),$$

where $s = (s_1, s_2)$, $t = (t_1, t_2)$ are two points of R_+^2. Our purpose is to show that for almost all sample functions of X, the set of local maxima is dense in R_+^2. In the one-parameter case, almost all Brownian sample functions are monotone in no interval. As a consequence of this property, it can be shown that for almost all sample functions, the set of local maxima is dense in R_+^1. A proof is given in Freedman (1971). However, this method of approach does not provide a suitable proof in the two-parameter case due to the complex behavior of the sample functions.

The notation of Orey and Pruitt (1973) will be used. For $s = (s_1, s_2)$ and $t = (t_1, t_2)$ with $s_i \leqslant t_i$, $i = 1, 2$, $\times_{i=1}^2 [s_i, t_i]$ is denoted by $\Delta(s, t)$ and $\Delta(t)$ in case $s = (0, 0)$. Let $s, t \in R_+^2$. The variance of $X(t) - X(s)$ can be verified to be the two dimensional Lebesgue measure of $S(s, t)$ where $S(s, t)$ is the symmetric difference between $\Delta(s)$ and $\Delta(t)$. We will often state that X has continuous sample functions, independent increments and denote the increments of X over $\Delta(s, t)$ by $X(\Delta(s, t))$. For an account of these properties and further information on X, consult Kitagawa (1951), Chentsov (1956), Yeh (1960, 1963, a, b), Delporte (1966), C. Park (1969), W. J. Park (1970), Zimmerman (1972).

DEFINITION. $X(\cdot, \omega)$ has a local maximum at s if there is an open set O containing s such that $O \subset R_+^2$ and $X(t, \omega) \leqslant X(s, \omega)$ for all $t \in O$.

LEMMA 1. *Let $s \in U$ where U denotes the unit cube and $C \subset U$ be a cube with center at s, sides parallel to coordinate axes and equal to a. Let I, ∂C be the interior and boundary of C. Then there exists a constant $\alpha > 0$ and an $\varepsilon > 0$ such that*

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$$P\left\{\sup_{t\in I} X(t) > \sup_{t\in\partial C} X(t) + a^{1/2}\right\} > \alpha,$$

whenever $a < \varepsilon$.

PROOF. Let (u_1, u_2) and (v_1, v_2) be the least and largest vertex of C. Then $s_1 = (u_1 + v_1)2^{-1}$, $s_2 = (u_2 + v_2)2^{-1}$. Define

$A_1 = \{X(s_1, u_2) - X(u_1, u_2) > 2a^{1/2}\}$,
$A_2 = \{X(s_1, u_2) - X(v_1, u_2) > 2a^{1/2}\}$,
$A_3 = \{X(u_1, s_2) - X(u_1, u_2) > 2a^{1/2}\}$,
$A_4 = \{X(u_1, s_2) - X(u_1, v_2) > 2a^{1/2}\}$.

Now A_1, A_2, A_3, A_4 are independent because X has independent increments. Pick C with $a < \min(s_1, s_2)$; then the variances of

$$X(s_1, u_2) - X(u_1, u_2), \qquad X(s_1, u_2) - X(v_1, u_2),$$
$$X(u_1, s_2) - X(u_1, u_2), \qquad X(u_1, s_2) - X(u_1, v_2)$$

are greater than $2^{-2}ca$, where c is any positive number smaller than s_1 and s_2. Therefore,

$$\begin{aligned} P(A_1A_2A_3A_4) &= P(A_1)P(A_2)P(A_3)P(A_4) \\ &> \gamma \quad \text{for some constant } \gamma. \end{aligned}$$

Let $A = A_1A_2A_3A_4$. Define

$$M = \left\{\sup_{t\in I} X(t) \geqslant \sup_{t\in\partial C} X(t) + a^{1/2}\right\}. \tag{1}$$

Let $\Delta(r, l) \subset U$ be any interval with both sides smaller than or equal to a. Clearly,

$$\begin{aligned} P(M) \geqslant\ & P\left(M/A \cap \left[\sup_{r,l\in U} |X(\Delta(r,l))| < a^{1/2}\right]\right) \\ & \cdot P\left(A \cap \left[\sup_{r,l\in U} |X(\Delta(r,l))| < a^{1/2}\right]\right). \end{aligned} \tag{2}$$

For each $t \in \partial C$, construct a point $p^t \in I$ as follows: If t is not a vertex of C, let p^t be the point obtained from the perpendicular projection of t onto the line passing through the center of C and parallel to the side of C on which t lies. If t is a vertex of C, pick p^t to be the center s of C. Now

$$P\left\{\left[\inf_{t\in\partial C}(X(p^t) - X(t)) > a^{1/2}\right] \Big/ A \cap \left[\sup_{r,l\in U} |X(\Delta(r,l))| < a^{1/2}\right]\right\} = 1.$$

It then follows that

$$P\left(M/A \cap \left[\sup_{r,l\in U} |X(\Delta(r,l))| < a^{1/2}\right]\right) = 1.$$

By (2), we obtain

$$P(M) \geqslant P\Big(A \cap \sup_{r,l \in U} |X(\Delta(r,l))| < a^{1/2}\Big).$$

But $P(A) > \gamma$ and $P(\sup_{r,l\in U}|X(\Delta(r,l))| < a^{1/2})$ goes to 1 by Theorem 2.1 of Orey and Pruitt (1973). The proof of the lemma is completed.

LEMMA 2. *Let $\{C_n\}$ be a sequence of cubes with common center s, sides parallel to coordinate axes and $C_n \supset C_{n+1}$ for all $n \geqslant 1$. Corresponding to each cube C_n, let M_n be defined as in* (1). *Suppose that the sides of C_n are small enough so that $P(M_n) > \alpha$ for all $n \geqslant 1$. Let the sides of C_n be a_n and assume $a_n \to 0$ as $n \to \infty$. Define*

$$S_n = \Big\{\sup_{t\in I_n} X(t) < \sup_{t\in \partial C_n} X(t) + a_n^{1/2}\Big\},$$

where I_n and ∂C_n are, respectively, the interior and boundary of C_n. Define

$$E_m = \bigcap_{i=1}^{m} S_i.$$

If μ is a constant < 1, then there exists a $k > m$ such that

$$P(M_k E_m) > \mu P(M_k) P(E_m).$$

PROOF. Pick C_k with $k > m$ and let $t \in R_+^2$. Since X has independent increments and continuous sample functions, it is possible to write $X(t) - X(s)$ as the sum of two independent random variables, say $H_k(t)$ and $J_k(t)$, chosen in such a way that $J_k(t)$ and M_k are independent; also $H_k(t)$ converges to zero almost surely as k goes to infinity. To do this, consider the symmetric difference $S(s,t)$ between $\Delta(s)$ and $\Delta(t)$. This symmetric difference is the union of two parts: $S(s,t) \cap S(t^k, s^k)$ and $S(s,t) \cap S(t^k,s^k)'$, where $'$ denotes the complement and t^k, s^k are the least and largest vertex of C_k. Each part can be considered as the union of at most two nonoverlapping intervals. It is now clear that $H_k(t)$ can be chosen so that it depends only on the increments over intervals that form $S(s,t) \cap S(t^k,s^k)$; and $J_k(t)$ can be chosen so as to depend only on increments over intervals that form $S(s,t) \cap [S(t^k,s^k)]'$. Since X has independent increments, $J_k(t)$ is independent of $H_k(t)$ and M_k. Pick $\varepsilon > 0$. Define

$$B = \Big[\sup_{t\in U} |H_k(t)| < \varepsilon\Big].$$

Clearly,

$$\begin{aligned}
P(M_k E_m) &\geqslant P(M_k E_m B) \\
&\geqslant P\Big(M_k \bigcap_{i=1}^{m} \Big\{\sup_{t\in I_i}(X_t - X_s) < \sup_{t\in\partial C_i}(X_t - X_s) + a_i^{1/2}\Big\} B\Big) \\
&\geqslant P\Big(M_k \bigcap_{i=1}^{m} \Big\{\sup_{t\in I_i}(J_k(t) + H_k(t)) < \sup_{t\in\partial C_i}(J_k(t) + H_k(t)) + a_i^{1/2}\Big\} \Big/ B\Big) P(B) \\
&\geqslant P\Big(M_k \bigcap_{i=1}^{m} \Big\{\sup_{t\in I_i} J_k(t) < \sup_{t\in\partial C_i} J_k(t) + a_i^{1/2} - 2\varepsilon\Big\} \Big/ B\Big) P(B).
\end{aligned}$$

But both B and M_k are independent of

$$\bigcap_{i=1}^{m}\left\{\sup_{t\in I_i} J_k(t) < \sup_{t\in\partial C_i} J_k(t) + a_i^{1/2}\right\}.$$

Therefore,

$$P(M_k E_m)$$
$$\geqslant P(M_k/B)P\left(\bigcap_{i=1}^{m}\left\{\sup_{t\in I_i} J_k(t) < \sup_{t\in\partial C_i} J_k(t) + a_i^{1/2} - 2\varepsilon\right\}\right)P(B).$$

But $P(M_k) > \alpha$ for all $k \geqslant 1$; and $P(B) \to 1$ as $k \to \infty$ since the sample functions of X are continuous. The proof is completed by picking k sufficiently large and ε sufficiently small.

THEOREM. *For almost all sample functions of X, the set of local maxima is dense in R_+^2.*

PROOF. Let $s \in R_+^2$ with $s_1 > 0$, $s_2 > 0$. By Lemmas 1 and 2, we can pick a sequence of cubes $\{C_n\}$ with center at s and $U \supset C_n \supset C_{n+1}$ for all $n \geqslant 1$ such that

$$P(M_{m+1}E_m) > \mu P(M_{m+1})P(E_m) \quad \text{for all } m \geqslant 1.$$

Now

$$P(E_{m+1}) = P(E_m M'_{m+1}) = P(E_m) - P(E_m M_{m+1})$$
$$< P(E_m)(1-\mu\alpha) < P(E_1)(1-\mu\alpha)^m.$$

Therefore $P(\bigcap_{m=1}^{\infty} S_m) = 0$. Since s is arbitrary and C_1 can be chosen arbitrarily small, the theorem follows.

REMARK. Other properties related to the local maxima of Brownian sample functions as presented in Freedman (1971) also hold in the two-parameter case. The proof of these properties follows the same line of argument used in the one-parameter case and therefore is not included here.

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