ON LIE ALGEBRAS WITH PRIMITIVE ENVELOPES, SUPPLEMENTS

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Abstract. Let $L$ be a finite dimensional Lie algebra over a field $k$ of characteristic zero, $U(L)$ its universal enveloping algebra and $Z(D(L))$ the center of the division ring of quotients of $U(L)$. A number of conditions on $L$ are each shown to be equivalent with the primitive of $U(L)$. Also, a formula is given for the transcendency degree of $Z(D(L))$ over $k$.

1. Introduction. The aim of this paper is to establish a necessary and sufficient condition on a finite dimensional Lie algebra $L$ over a field $k$ in order that its universal enveloping algebra $U(L)$ is primitive. This settles a problem raised by Professor Jacobson [6, p. 23]. We may restrict ourselves to the case where $k$ is of characteristic zero, since in characteristic $p \neq 0$, $U(L)$ is not primitive unless $L = 0$ [6, p. 255]. On the other hand, $k$ is not assumed algebraically closed throughout the paper. Let $D(L)$ be the division ring of quotients of $U(L)$, $Z(D(L))$ its center. Let $G \subset \text{Aut } L$ be the smallest algebraic group whose Lie algebra $L(G)$ contains $\text{ad } L$ (i.e. $L(G)$ is the algebraic hull of $\text{ad } L$ in $\text{End } L$). $G$ is called the adjoint algebraic group of $L$. For each linear functional $f \in L^*$ we define $L[f]$ to be the collection of all $x \in L$ such that $f([E,x]) = 0$ for all $E \in L(G)$. $L[f]$ is a Lie subalgebra of $L$ containing the center of $L$. One verifies that $L[f]$ is an ideal of $L(f)$, where $L(f)$ is the radical of the alternating bilinear form $(x,y) \mapsto f([x,y])$ defined on $L$. Clearly $L[f] = L(f)$ if $L$ is ad-algebraic. Furthermore, let $K(L)$ be the quotient field of the symmetric algebra $S(L)$, $K(L)^I$ the subfield of invariants of $K(L)$.

We can now state the main result.

Theorem. The following conditions are equivalent:

1. $L[f] = 0$ for some $f \in L^*$.
2. $G$ admits an open dense orbit in $L^*$ for its contragredient action on $L^*$.
4. $Z(D(L)) = k$.
5. $U(L)$ is primitive.

The proof uses some striking properties of the Dixmier-Duflo map [3, pp. 314–320] as well as some earlier results on the subject [7]. Finally, we shall verify that the number $t = \min_{f \in L^*} \dim L[f]$ is equal to the transcendency degree.
degree of $Z(D(L))$ over $k$. This follows directly from the isomorphism that exists between $K(L)^I$ and $Z(D(L))$ in the algebraically closed case [8].

2. It is understood that we consider the Zariski topology on $L^*$. We denote by $O(f)$ the orbit of $f \in L^*$ under the contragredient action of $G$ on $L^*$. $O(f)$ is irreducible (since $G$ is irreducible) and open in its closure [1, p. 98]. Following Dixmier we call $r = \min_{f \in L^*} \dim L(f)$ the index of $L$ and $f \in L^*$ is called regular if $\dim L(f) = r$ [3, p. 51]. It is well known that the set $L^*_\text{reg}$ of all regular linear functionals is an open dense $G$-stable subset of $L^*$. A similar property, concerning the Lie subalgebras $L[f]$, is obtained in the following.

**Lemma 1.** For all $f \in L^*$ we have $\dim L[f] + \dim O(f) = \dim L$. Moreover, the collection $\Omega$ of all $f \in L^*$ such that $\dim L[f] = t$ is an open dense, $G$-stable subset of $L^*$ ($t = \min_{f \in L^*} \dim L[f]$).

**Proof.** If $(x_1, \ldots, x_n)$ is a basis for $L$ and $(E_1, \ldots, E_m)$ a basis for $L(G)$, then it is easily seen that

$$\dim L[f] = \dim L - \text{rank } (f(E_i x_j)_{ij}).$$

On the other hand, the stabilizer $S(f)$ of $f \in L^*$ is a closed subgroup of $G$ and

$$\dim O(f) = \dim G - \dim S(f) = \dim L(G) - \dim L(S(f)),$$

where $L(S(f))$, being the Lie algebra of $S(f)$, is the set of all $E \in L(G)$ such that $f \circ E = 0$. By considering the bilinear map $L(G) \times L \rightarrow k$ sending $(E, x)$ into $f(Ex)$ we observe that

$$\dim L(S(f)) = \dim L(G) - \text{rank } (f(E_i x_j)_{ij}).$$

Hence

$$\dim O(f) = \text{rank } (f(E_i x_j)_{ij}) = \dim L - \dim L[f].$$

This takes care of the first part of the lemma.

In particular,

$$\max_{f \in L^*} \dim O(f) = \text{rank}_{K(L)} ((E_i x_j)_{ij}) = n - t.$$

Thus $\Omega = \{ f \in L^* | \text{rank } (f(E_i x_j)) = n - t \}$ and is therefore an open dense subset of $L^*$. Being the union of all orbits of maximum dimension, $\Omega$ is also $G$-stable.

The following is a result due to Gabriel [3, p. 159].

**Theorem.** Let $I$ be a two-sided ideal of $U(L)$. Then the following conditions are equivalent:

(i) $I$ is absolutely primitive (i.e. $I \otimes k'$ is primitive in $U(L \otimes k')$ for every field extension $k'$ of $k$).

(ii) There exists an algebraically closed extension $k'$ of $k$ such that $I \otimes k'$ is primitive in $U(L \otimes k')$.

(iii) $I$ is primitive and the center of the ring of quotients of $U(L)/I$ reduces to $k$. 
We are now in a position to prove the main criterion.

**Theorem 1.** Let $L$ be a Lie algebra over $k$. Then the following conditions are equivalent:

1. $L[f] = 0$ for some $f \in L^*$.
2. $G$ admits an open dense orbit in $L^*$ for its contragredient action on $L^*$.
3. $K(L)^f = k$.
4. $Z(D(L)) = K$.
5. $U(L)$ is primitive.

**Proof.** The equivalence of (1), (3) and (4) has already been shown in [7], as well as the implication (5) $\Rightarrow$ (1). Let us now verify (1) $\iff$ (2). Assume $L[f] = 0$ for a suitable $f \in L^*$. Then Lemma 1 implies that $\dim O(f) = n = \dim L^*$. Consequently $O(f)$ is dense in $L^*$. It is even open in $L^*$ since $O(f)$ is open in its closure. Conversely, if $O(f)$ is open and dense in $L^*$ for some $f \in L^*$, then $\dim O(f) = n$ which forces $L[f] = 0$ (Lemma 1). Moreover, such an orbit is evidently unique (if $O(h), h \in L^*$, is also open, then $O(f) \cap O(h) \neq \emptyset$ and thus $O(f) = O(h)$). Since $O(f)$ is the only orbit of maximum dimension, it follows that $O(f) = \Omega$. Therefore $\Omega \cap L^*_\text{reg} \neq \emptyset$ implies that $\Omega \subset L^*_\text{reg}$.

(1) $\Rightarrow$ (5). Let $k'$ be the algebraic closure of $k$ and put $L' = L \otimes k'$. Denote by $\text{Prim}(U(L'))$ the set of all primitive ideals of $U(L')$, endowed with the Jacobson topology [6, p. 203] and let $J: L^*_\text{reg} \rightarrow \text{Prim}(U(L'))$ be the Dixmier-Duflo map which assigns to each regular functional $f \in L^*$ a primitive ideal $J(f)$ of $U(L')$. $J$ is known to be continuous [3, p. 317] and constant on the orbits lying in $L^*_\text{reg}$ (i.e. $J(g \cdot f) = J(f)$ for all $g \in G'$, $G'$ being the algebraic adjoint of $L'$) [3, p. 84], [8, p. 394]. Furthermore, if $Q \subset L^*_\text{reg}$ is dense in $L^*$ then $\cap_{f \in Q} J(f) = 0$ [3, p. 320].

In carrying out the proof of Lemma 1 we came across the formula

$$t = \min_{f \in L} \dim L[f] = \dim L - \text{rank}_{K(L)}((E_i x_j)_{ij})$$

whenever $\{E_1, \ldots, E_m\}$ is a basis for $L(G)$ and $\{x_1, \ldots, x_n\}$ a basis for $L$. Clearly this number $t$ remains unchanged under extension of the base field $k$. So, if $L$ satisfies (1) (i.e. $t = 0$) the same holds for $L'$. Then the foregoing observation shows that there exists an orbit $\Omega' \subset L^*_\text{reg}$ which is open dense in $L^*$. Choose $h \in \Omega'$. Application of the properties of the map $J$ mentioned above gives

$$J(h) = \bigcap_{f \in \Omega'} J(f) = 0.$$

Hence the ideal (0) is primitive in $U(L')$ and by Gabriel's theorem also in $U(L)$. This completes the proof.

**Remark.** Because of this theorem, all examples of Lie algebra we have listed in [7] have primitive envelopes, even without the requirement that the base field $k$ is algebraically closed.

Probably the most interesting class of Lie algebras satisfying the conditions of Theorem 1 is formed by the Lie algebras of index 0, partly because they include all ad-algebraic Lie algebras enjoying these conditions. If $L$ is of index 0, it admits a linear functional $f \in L^*$ such that the alternating bilinear form
on $L$ sending $(x, y)$ into $f([x, y])$ is nondegenerate, a situation reminiscent of Frobenius algebras in the associative case. In the study of these so called Frobenius Lie algebras, the Lie algebra of all $n \times n$ matrices with entries in $k$ and with last row equal to zero seems to play a significant role. It is an ad-algebraic Frobenius Lie algebra satisfying the Gelfand-Kirillov conjecture [4], [7]. However, not all Frobenius Lie algebras are ad-algebraic (example b(iii) of [7, p. 497] is not even almost algebraic).

**Proposition.** Let $L$ be a (finite dimensional) Lie algebra over $k$. If $f, f' \in L^*$ are such that $L[f] = 0 = L[f']$, then $f' = g \cdot f$ for some $g \in G$, $G$ being the adjoint algebraic group of $L$. In particular, in a Frobenius Lie algebra any two regular linear functionals are conjugate by an element of the adjoint algebraic group.

**Proof.** We know from the proof of Theorem 1 that the set $\Omega$ of all $f \in L^*$ such that $L[f] = 0$ is an orbit under the action of $G$ on $L^*$.

3. Next we want to establish a formula for the transcendency degree $\text{tr.deg}_k (Z(D(L)))$ of the center $Z(D(L))$ over $k$. For this task we need to recall the following preliminary material.

Let $s$ be the canonical linear isomorphism of $S(L)$ onto $U(L)$, which maps each product $y_1 \cdots y_q$, $y_i \in L$, into $(1/q!) \sum p_{y_1} \cdots y_{pq}$ where $p$ ranges over all permutations of $\{1, \ldots, q\}$. Let $\{x_1, \ldots, x_n\}$ be a basis of $L$ and $\{E_1, \ldots, E_m\}$ a basis for $L(G)$. Then $S(L) \cong k[x_1, \ldots, x_n]$ is the direct sum of the subspaces $S^q$ of homogeneous polynomials of degree $q$. On the other hand, let $U_q$, $q \geq 0$, be the family of subspaces of $U(L)$ which forms the usual increasing filtration of $U(L)$. The associated graded algebra is isomorphic to $S(L)$ by the Poincaré-Birkhoff-Witt theorem. The elements $u \in U_q \setminus U_{q-1}$ are said to be of degree $q$ and $[u] = u$ mod $U_{q-1}$ is called the leading term of $u$. All nonzero elements $u, v \in U(L)$ satisfy $[uv] = [u][v]$ and $\deg (uv) = \deg (u) + \deg (v)$. If $x = x_0 + \cdots + x_0$, $x \neq 0$, is the decomposition of $x \in S(L)$ into homogeneous components $(x_i \in S^i)$ then we notice that $[s(x)] = x_q$. Every $E \in \text{ad} L$ acts as a derivation in both $K(L)$ and $D(L)$, leaving stable the subspaces $S^q$ and $U_q$, and commutes with $s$ (i.e. $Es(x) = s(Ex)$ for all $x \in S(L)$).

In order to proceed we require the following lemmas.

**Lemma 2.** $K(L)^I$ is generated as a field by elements of the form $xy^{-1} \in K(L)^I$, $y \neq 0$, where $x$ and $y$ are homogeneous semi-invariants, i.e. $x \in S^i$, $y \in S^j$ for some $i, j \in \mathbb{N}$ and we can find a $\lambda \in (\text{ad} L)^*$ such that $Ex = \lambda(E)x$, $Ey = \lambda(E)y$ for all $E \in \text{ad} L$.

**Proof.** Let $u \in K(L)^I$. We may write $u = xy^{-1}$, $y \neq 0$, where $x, y \in S(L)$ are relatively prime. A standard argument shows that there is a $\lambda \in (\text{ad} L)^*$ such that $Ex = \lambda(E)x$, $Ey = \lambda(E)y$ for all $E \in \text{ad} L$. Let $x = x_p + \cdots + x_0$, $y = y_q + \cdots + y_0$ be the decomposition into homogeneous components $(x_i \in S^i, y_j \in S^j)$. Since each $E \in \text{ad} L$ maps each $S^i$ into itself we see that $Ex = Ex_0 + \cdots + Ex_0$ is the corresponding decomposition of $Ex$. It follows that $Ex_i = \lambda(E)x_i$, and similarly $Ey_j = \lambda(E)y_j$ for all $i, j$ and for all $E \in \text{ad} L$. Finally,
\[
\begin{align*}
    u &= x y^{-1} = \sum_i x_i y_i^{-1} = \sum_i \left( \sum_j y_j x_i^{-1} \right)^{-1}
\end{align*}
\]

(only those indices \(i\) are considered for which \(x_i \neq 0\) where each \(y_j x_i^{-1} \in K(L)\) satisfies the requirements of the lemma.

**Lemma 3.** \(\text{tr deg}_k (K(L)^I) \leq \text{tr deg}_k (Z(D(L))).\)

**Proof.** The previous lemma guarantees that we can single out a transcendency basis for \(K(L)^I\) of the form \(x_1 y_1^{-1}, \ldots, x_t y_t^{-1}, y_i \neq 0\), where all \(x_i, y_i \in S(L)\) are homogeneous semi-invariants. Put \(u_i = s(x_i), v_i = s(y_i)\) and \(z_i = u_i v_i^{-1}\). We observe that for all \(E \in \text{ad} \ L, E u_i = E s(x_i) = \lambda(E) s(x_i) = \lambda(E) u_i\) and similarly \(E v_i = \lambda(E) v_i\). Consequently, \(z_i \in Z(D(L))\) since
\[
    E z_i = E (u_i v_i^{-1}) = (E u_i - u_i v_i^{-1} E v_i) v_i^{-1} = (\lambda(E) u_i - u_i v_i^{-1} \lambda(E) v_i) v_i^{-1} = 0 \quad \text{for all } E \in \text{ad} \ L.
\]

Clearly, it suffices to show that \(z_1, \ldots, z_t\) are algebraically independent over \(k\). Suppose we can find some \(a_q \in k\), not all zero \((q = (q_1, \ldots, q_t))\) such that \(\sum_q a_q z_1^{q_1} \cdots z_t^{q_t} = 0\). Let \(m_i\) be the largest exponent of \(z_i\) that appears nontrivially in this sum. Since \(u_i\) and \(v_i\) commute with each other we obtain, after multiplication with \(v_i^{m_i-1} \cdots v_1^{m_1-1}\), that
\[
    \sum_{q \in Q} a_q [u_1]^{q_1} [v_1]^{m_1-q_1} \cdots [u_t]^{q_t} [v_t]^{m_t-q_t} = 0.
\]

Let \(m\) be the largest degree (as defined in the preliminaries) of all monomials appearing nontrivially in this sum and let \(Q\) be the set of all \(q\)'s with \(a_q \neq 0\) and corresponding with the monomials of degree \(m\). Then it follows that
\[
    \sum_{q \in Q} a_q [u_1]^{q_1} [v_1]^{m_1-q_1} \cdots [u_t]^{q_t} [v_t]^{m_t-q_t} = 0.
\]

After dividing by \(v_1^{m_1} \cdots v_t^{m_t}\) and taking into account that \([u_i] = [s(x_i)] = x_i\) and \([v_i] = [s(y_i)] = y_i\) we conclude that \(\sum_{q \in Q} a_q (x_1 y_1^{-1})^{q_1} \cdots (x_t y_t^{-1})^{q_t} = 0\) which contradicts our original assumption.

**Lemma 4.** Let \(k'\) be an extension field of \(k\) and put \(L' = L \otimes k'\). Then \(\text{tr deg}_k (Z(D(L))) \leq \text{tr deg}_{k'} (Z(D(L'))).

**Proof.** The identification of \(U(L) \otimes k'\) with \(U(L')\) results in an imbedding of \(D(L) \otimes k'\) into \(D(L')\) and thus \(D(L)\) and \(k'\) are linearly disjoint in \(D(L')\). Therefore we may consider \(Z(D(L)) \otimes k' \subset Z(D(L'))\). Suppose \(z_1, \ldots, z_p \in Z(D(L))\) are algebraically independent over \(k\). This means that the monomials \(z_1^{n_1} \cdots z_p^{n_p}, n_i \in \mathbb{N}\), are linearly independent over \(k\). Hence \(z_1^{n_1} \cdots z_p^{n_p} \otimes 1, n_i \in \mathbb{N}\), are linearly independent over \(k'\). This implies that \(z_1 \otimes 1, \ldots, z_p \otimes 1\) are algebraically independent over \(k'\). The result then follows immediately.

**Theorem 2.** Let \(L\) be a Lie algebra over \(k\), \(G\) its adjoint algebraic group acting on \(L^*\) and \(M\) the largest dimension of all orbits in \(L^*\). Then \(Z(D(L))\) and \((K(L))^I\) have the same transcendency degree over \(k\), equal to the number \(t = \min_{f \in L^*} \dim L[f] = \dim L - M\).
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Proof. Let \( k' \) be the algebraic closure of \( k \) and put \( L' = L \otimes k' \). Then the fields \( Z(D(L')) \) and \( K(L')^I \) are \( k' \)-isomorphic [8, p. 401]. This combined with the preceding lemmas yields

\[
\text{tr deg}_k (K(L)^I) \leq \text{tr deg}_k (Z(D(L))) \leq \text{tr deg}_k (Z(D(L'))) = \text{tr deg}_{k'} (K(L')^I).
\]

On the other hand, \( \text{tr deg}_k (K(L)^I) = \text{tr deg}_{k'} (K(L')^I) \). Indeed, we know that

\[
\text{tr deg}_k (K(L)^I) = \dim L - \text{rank}_{K(L)} ((E_i x_j)_{ij}) \quad [7],
\]

which we have seen (in the proof of Theorem 1) to be equal to \( t = \min_{f \in L^*} \dim L[f] = \dim L - M \) and which does not change under field extension. Hence, we may conclude that

\[
\text{tr deg}_k (Z(D(L))) = \text{tr deg}_k (K(L)^I) = t.
\]

Remark. In case \( L \) is ad-algebraic the formula we came across in the preceding discussion simplifies to

\[
\text{tr deg}_k (Z(D(L))) = \dim L - \text{rank}_{K(L)} ([x_i, x_j])
\]

which is now equal to the index of \( L \).

References

An extensive reference list can be found in Dixmier's book on enveloping algebras [3].


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