A SUBCLASS OF ANHARMONIC OSCILLATORS WHOSE EIGENFUNCTIONS HAVE NO RECURRENCE RELATIONS[1]

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ABSTRACT. The equation $w''(z) + (\lambda - p(z))w(z) = 0$ $(z \in \mathbf{C}, \lambda \in \mathbf{R})$ with a fixed $p(z) = a_{2m}z^{2m} + a_{2m-2}z^{2m-2} + \cdots + a_2 z^2$ $(a_i \geqslant 0 \; \forall i, a_{2m} > 0)$, possesses a set of solutions $\{\psi_n(z)\}_{n=0}^{\infty}$ (with associated $\{\lambda_n\}_{n=0}^{\infty}$) which form a complete orthonormal set for $L^2(-\infty, \infty)$ (as a real space). Here it is shown that any $g(z)\psi_N^{(q)}(z)$ (fixed N, $q \in \mathbf{N}$, $g \neq 0$ a polynomial with deg $(g) \geqslant 1$ when $q = 0$) cannot be expressed as a finite linear combination of $\{\psi_n(z)\}_{n=0}^{\infty}$, when deg$(p)$ is a multiple of 4. It is well known that $g(z)\psi_N^{(q)}(z)$ can always be expressed as such when $p(z) = z^2$ (the Hermite case).

I. **Introduction.** For a fixed $p(z) = a_{2m}z^{2m} + a_{2m-2}z^{2m-2} + \cdots + a_2 z^2$ where $a_i \geqslant 0 \; \forall i$, $a_{2m} > 0$, the equation

$$(1) \qquad w''(z) + (\lambda - p(z))w(z) = 0, \quad \lambda \in \mathbf{R}, z \in \mathbf{C},$$

has a complete orthonormal set of solutions $\{\psi_n(z)\}_{n=0}^{\infty}$ (eigenfunctions) for $L^2(-\infty, \infty)$ (as a real space), with corresponding discrete positive eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$. ψ_n is even or odd as n is even or odd [9, p. 172], and any L^2 solution of (1) is a multiple of one of these eigenfunctions. [9] has an extensive theory on these matters.

Equation (1) with $p(z) = z^2$, yields

$$\psi_n(z) = K_n H_n(z)\exp(-z^2/2) \quad (\lambda_n = 2n + 1)$$

where H_n is the Hermite polynomial of degree n. The well-known recurrence formulas for Hermite polynomials allow the expression of any $g(z)\psi_N^{(q)}(z)$ (g a polynomial; fixed $q, N \in \mathbf{N}$) as a finite linear combination of $\{\psi_n\}_{n=0}^{\infty}$. This is shown briefly. The purpose of this paper is to show that when deg $(p) = 2m$ is a multiple of 4, any such $g\psi_N^{(q)}$ ($g \neq 0$ and deg $(g) \geqslant 1$ when $q = 0$) cannot be expressed similarly. J. Zeitlin [11] proved this for the special case deg $(p) = 4$, by Lie algebra calculations. The crux of the argument here is imaginary

axis behavior: $\psi_n(iy) \to 0$ but $\psi_n{'}(iy) \nrightarrow 0$ as $|y| \to \infty$ (all n). This will be discussed in more detail later.

Remaining as an open question is what happens in the cases $m \geqslant 3$ odd. It would seem that, like here, the answers are negative. A discussion of this is in [5, pp. 31–32].

Our result is of the following interest. In many cases where the eigenfunctions of Sturm-Liouville operators are "known" special functions, there is an associated Lie algebra $\mathcal{G}$ of differential operators ($[A, B] = AB - BA$) whose action on the eigenfunctions yields expressions for $\psi_N^{(q)}$, recurrence relations, and other things. For the operator $H = (d/dz)^2 - p(z)$, $\mathcal{G}$ is the $(m + 1)$-step nilpotent Lie algebra with basis $\{d/dz, i1, iz, iz^2, \ldots, iz^m\}$, and our result shows that, for m even, it cannot give *finite* recurrence relations. It certainly answers R. Goodman's particular question [10, 4.4.7] of whether for fixed N: $\mathcal{G} \cdot \psi_N$ $\subset$ linear span $\{\psi_n\}_{n=0}^{\infty}$?

Generalizations can readily be made. Consider the Lie algebraic fact that if T is any differential operator with polynomial coefficients, then $T\psi_N$ is expressible as a finite sum of eigenfunctions $\Leftrightarrow [H, T]\psi_N$ is also expressible. It follows that, if $T = g(d/dz)^q$ as above and ad $H(T) = [H, T]$, then $(\text{ad } H)^n$ $\cdot (T)$ has the nonrecursive property for all $n \in \mathbf{N}$. Furthermore, the proof used on T will generalize to show that the linear span of $\{g(d/dz)^q \neq \text{constant}:$ fixed polynomial $g, q \in \mathbf{N}\}$ also has the nonrecursive property. Obviously, for fixed q and varying g, the resulting linear space has the property.

Explaining the title of the paper, $(d/dx)^2 - q(x)$ is an anharmonic oscillator when q is a polynomial of even degree $\geqslant 4$, with positive leading coefficient. Anharmonic refers to the fact that the eigenvalues are not in an arithmetic progression. There has been intensive recent study of these operators for applications to quantum field theory. See, for example, [2], [3], [8]. Other recent work on these operators is in [6], [7].

II. **The eigenfunctions when** $p(z) = z^2$. In this case, $\psi_n(z) = K_n H_n(z)$ $\cdot \exp(-z^2/2)$ where H_n is the Hermite polynomial of degree n. We have the following recurrence relations for Hermite polynomials [1, p. 162]:

LEMMA 1. (i) $H_n{'}(z) = 2nH_{n-1}(z)$ *for* $n \geqslant 1$; $H_0{'}(z) = 0$.
(ii) $H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z)$ *for* $n \geqslant 1$; $H_1(z) = 2zH_0(z)$.

PROPOSITION 2. $\forall N, q \in \mathbf{N}$ *and polynomial* g: $g\psi_N^{(q)}$ *is always a finite linear combination of* $\{\psi_n\}_{n=0}^{\infty}$.

PROOF. From Lemma 1, we get

$$\psi_N{'}(z) = B_N \psi_{N-1}(z) + C_N \psi_{N+1}(z), \qquad N \geqslant 1,$$

$$= D_N \psi_1(z), \qquad\qquad\qquad N = 0.$$

$$z\psi_N(z) = b_N \psi_{N+1}(z) + c_N \psi_{N-1}(z), \qquad N \geqslant 1,$$

$$= d_N \psi_1(z), \qquad\qquad\qquad N = 0.$$

So $\forall k, q \in \mathbf{N}$, $\psi_N^{(q)}(z)$ and $z^k \psi_N(z)$ are finite linear combinations of $\{\psi_n\}_{n=0}^{\infty}$, and the assertion follows.

REMARK. For another proof of Proposition 2, see [5, p. 29].

III. **The eigenfunctions when m is even.** In this case, on the imaginary axis: $\psi_N(iy) \to 0$ as $|y| \to \infty$ $\forall$ fixed N (throughout this section, N will be fixed). The argument, the details of which can be found in [5], will now be outlined.

Define $\phi: \mathbf{R} \to \mathbf{R}$ by

(2)
$$\phi(y) = \psi_N(iy) \qquad \text{if } N \text{ is even,}$$
$$= i\psi_N(iy) \qquad \text{if } N \text{ is odd.}$$

Then equation (1) gives

(3)
$$\phi''(y) + (p(iy) - \lambda_N)\phi(y) = 0.$$

Pick y_0 so that $p(iy) > \lambda_N$ for $y \geqslant y_0$. Then for $y \geqslant y_0$, the transformations (WKB)

(4)
$$\xi(y) = \int_{y_0}^{y} (p(iu) - \lambda_N)^{1/2} \, du, \quad \eta(y) + (p(iy) - \lambda_N)^{1/4} \phi(y)$$

take equation (3) into

(5)
$$\frac{d^2\eta}{d\xi^2} + \eta(y) + \eta(y)\left[\frac{p''(iy)}{4(p(iy) - \lambda_N)^2} - \frac{5(p'(iy))^2}{16(p(iy) - \lambda_N)^3}\right] = 0.$$

Note that the last coefficient of η goes to zero as $|y| \to \infty$. All solutions $\eta = \theta(\xi)$ of (5) satisfy the integral equation

$$\theta(\xi) = \theta(0) \cos \xi + \theta'(0) \sin \xi - \int_0^{\xi} \sin (\xi - T) R(t)\theta(T) \, dT$$

where

$$T = \xi(t) \quad \text{and} \quad R(t) = \frac{p''(it)}{4(p(it) - \lambda_N)^2} - \frac{5(p'(it))^2}{16(p(it) - \lambda_N)^3}.$$

From this integral equation it can be deduced that: for A, B constants (not both zero),

(6) $\phi(y) = (p(iy) - \lambda_N)^{-1/4}[A \cos \xi(y) + B \sin \xi(y) + o(1)]$ as $y \to \infty$.

This method of deriving (6) from a general equation, similar to (3), can be found in 5.8 and 5.10 of [9].

(6) shows, among other things, that $\psi_N \to 0$ along the imaginary axis (by (2)). Curiously, $\psi_N' \nrightarrow 0$ along the imaginary axis. Although this fact comes out in the course of obtaining (6) [5, p. 27], below we give an elementary proof which uses none of the machinery developed to get to (6).

Because $p(iy) \to \infty$ as $|y| \to \infty$, (2) and (3) imply that ψ_N has an infinite

number of pure imaginary zeros (for $y \geqslant y_0$, compare (3) with $f''(y) + f(y)$ $= 0$ and use Sturm's theorem on p. 107 of [9]). This oscillatory character makes (6) believable.

DEFINITION 3. Let $\{iA_\nu\}_{\nu=1}^{\infty}$ be the pure imaginary zeros of ψ_N with $A_\nu > 0$ (ordered by increasing modulus).

LEMMA 4. *Choose* $A > 0$ *so that* $ip'(iy) > 0$ *for* $y \geqslant A$. *Then for* $A_\nu > A$; *that is, for* $\nu \geqslant \nu_0$:

$$|\psi_N'(iA_\nu)|^2 > (p(iA) - \lambda_N)|\psi_N(iA)|^2 + |\psi_N'(iA)|^2.$$

PROOF. For $\nu \geqslant \nu_0$, integration by parts gives

$$\int_A^{A_\nu} \phi''(y)\phi'(y)\,dy = \int_A^{A_\nu} (\lambda_N - p(iy))\phi(y)\phi'(y)\,dy,$$

$$(\phi'(A_\nu))^2 - (\phi'(A))^2 = (\phi(A))^2(p(iA) - \lambda_N) + \int_A^{A_\nu} ip'(iy)(\phi(y))^2\,dy$$

$$> (\phi(A))^2(p(iA) - \lambda_N)$$

and the result follows from (2).

COROLLARY 5. $|\psi_N'(iA_\nu)| > |\psi_N'(iA_{\nu-1})|$ $\forall \nu \geqslant \nu_0 + 1$.

PROOF. Set $A = A_{\nu-1}$ in Lemma 4.

THEOREM 6. $\forall q, N \in \mathbf{N}$ *and polynomial* $g \not\equiv 0$ (deg $(g) \geqslant 1$ *when* $q = 0$): $g\,\psi_N^{(q)}$ *cannot be expressed as a finite linear combination of* $\{\psi_n\}_{n=0}^{\infty}$.

PROOF. Assume the contrary. Then an induction argument on the derivatives of $\psi_N^{(2)}(z) = (p(z) - \lambda_N)\psi_N(z)$ gives $\forall q \geqslant 1$ $(q \neq 2)$:

$$g(z)\psi_N^{(q)}(z) = g_1(z)\psi_N(z) + g_2(z)\psi_N'(z) = \sum_{k=0}^{K} c_k \psi_{n_k}(z), \qquad z \in \mathbf{C},$$

where g_1, g_2 are polynomials with $g_2 \not\equiv 0$. Therefore

$$\lim_{\nu \to \infty} g_2(iA_\nu)\psi_N'(iA_\nu) = \lim_{\nu \to \infty} \sum_{k=0}^{K} c_k \psi_{n_k}(iA_\nu).$$

Since every eigenfunction goes to zero along the imaginary axis ((6) with (2)) $\Rightarrow$ the right side above is zero. But by Corollary 5, the left side is not zero; hence a contradiction.

Now suppose $g(z)\psi_N(z) = \sum_{k=0}^{K} c_k \psi_{n_k}(z)$ where deg $(g) \geqslant 1$. Differentiating twice:

$$g''(z)\psi_N(z) + 2g'(z)\psi_N'(z) + (g(z)\psi_N(z))(p(z) - \lambda_N)$$

$$= \sum_{k=0}^{K} c_k(p(z) - \lambda_{n_k})\psi_{n_k}(z),$$

$$g''(z)\psi_N(z) + 2g'(z)\psi_N'(z) = \sum_{k=0}^{K} c_k(\lambda_N - \lambda_{n_k})\psi_{n_k}(z).$$

Again we get a contradiction by the previous argument.

Finally, if $g(z)\psi_N^{(2)}(z) = g(z)(p(z) - \lambda_N)\psi_N(z) = \sum_{k=0}^{K} c_k \psi_{n_k}(z)$, this would contradict what was just proven.

The proof is now complete.

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