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AN INEQUALITY FOR POSITIVE DEFINITE VOLTERRA KERNELS

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ABSTRACT. We deduce an inequality satisfied by certain positive definite Volterra kernels. This inequality yields a new theorem on the asymptotic behavior of the bounded solutions of a Volterra equation.

1. **Introduction.** Fourier transform methods have recently been used to get very sharp results on the asymptotic behavior of the bounded solutions of the nonlinear Volterra equation

$$x'(t) + \int_{[0,t]} g(x(t-s))\,d\mu(s) = f(t) \qquad (t \in R^+);\ x(0) = x_0; \tag{1}$$

see e.g. [1], [3], [5], [6] and [7]–[10] (we write R^+ for the interval $[0, \infty)$). This approach is possible whenever the function f is integrable and the kernel μ is a positive definite measure, i.e. the distribution Fourier transform $\hat{\mu}$ of μ has a nonnegative real part (no integrability of f is required in [3]). The treatment in [1], [5] and [6] is based on the notion of a "strongly positive definite" kernel: There exists $\varepsilon > 0$ such that $\operatorname{Re} \hat{\mu}(\omega) \geqslant \varepsilon(1 + \omega^2)^{-1}$ $(\omega \in R)$ (interpret the inequality in the distribution sense if μ does not have a classical Fourier transform). This requirement has been relaxed in [7] and [8] to "strict positive definiteness": $\operatorname{Re} \hat{\mu}$ is strictly positive everywhere in the appropriate sense. The question how zeros of $\operatorname{Re} \hat{\mu}$ affect the asymptotic behavior of the solutions of (1) is studied in [9] and [10].

The purpose of this paper is to present yet another Fourier transform condition, which overlaps the notion of strict positive definiteness. We consider kernels μ with a finite total variation satisfying

(2) *There exists* $\alpha \geqslant 0$ *such that*

$$\alpha \operatorname{Re} \hat{\mu}(\omega) \geqslant |\hat{\mu}(\omega)|^2 \ (\omega \in R).$$

This assumption is clearly stronger than positive definiteness, but it does not imply strict positive definiteness since $\operatorname{Re} \hat{\mu}(\omega) = 0$ is possible in some cases (a specific example is given following Theorem 2 below). On the other hand, not all strictly positive definite kernels satisfy (2).

The usefulness of (2) is due to the fact that it yields

LEMMA 1. *Let μ be a finite (real) Borel measure on R^+, and suppose that* (2)

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holds, where $\hat{\mu}(\omega) = \int_{R^+} e^{-i\omega t}\, d\mu(t)$ $(\omega \in R)$. *Then*

$$\text{(3)}\qquad \int_{[0,T]} \left| \int_{[0,t]} \varphi(t-s)\, d\mu(s) \right|^2 dt \leqslant \alpha \int_{[0,T]} \varphi(t) \int_{[0,t]} \varphi(t-s)\, d\mu(s)\, dt$$

for every $T \in R^+$, *and for every* (*real*) $\varphi \in C[0, T]$.

Inequality (3) is the key ingredient in the proof of

THEOREM 1. (i) *Let* $g \in C(R)$, $f \in L^1(R^+)$, *and let* μ *satisfy the assumption of Lemma* 1. *Moreover, suppose that* x *is a bounded solution of* (1) *on* R^+. *Then* $x' - f \in L^2(R^+)$.

(ii) *Let the assumption of* (i) *hold. In addition, suppose that* $\int_{R^+} |\mu([0,s])|\, ds < \infty$. *Then*

$$\lim_{t\to\infty} \left\{ x(t) + g(x(t)) \int_{R^+} \mu([0,s])\, ds \right\} = x_0 + \int_{R^+} f(s)\, ds.$$

(iii) *Let the assumption of* (i) *hold. In addition, suppose that*

$$\sup_{T>0} \int_{[0,T]} \mu([0,s])\, ds = \infty.$$

Then $g(x(t)) \to 0$ $(t \to \infty)$.

Theorem 1 motivates us to study the following question: Which classical kernels satisfy (2)?

THEOREM 2. *Condition* (2) *is satisfied in each of the following cases*:

(i) $b(t) = \mu([0,t])$ *is nonnegative and nonincreasing on* R^+.

(ii) $d\mu(t) = a(t)\, dt$ $(t \in R^+)$, *where* $a \in L^1(R^+) \cap BV(R^+)$ *is strongly positive definite, i.e. there exists* $\varepsilon > 0$ *such that* $\int_{R^+} \cos(\omega t) a(t)\, dt \geqslant \varepsilon(1 + \omega^2)^{-1}$ $(\omega \in R)$.

(iii) $d\mu(t) = a(t)\, dt$ $(t \in R^+)$, *where* a *and* $-a'$ *are nonnegative and convex on* $(0, \infty)$, *and* $a \in L^1(R^+)$.

Using Theorem 2(i) we can give a nontrivial example of a kernel which satisfies (2), but which is not strictly positive definite. Take $b(t) = 1$ $(0 \leqslant t \leqslant 1)$, $b(t) = 0$ $(t > 1)$, and define μ as in (i). Then (2) holds. However, calculating $\operatorname{Re} \hat{\mu}(\omega) = 1 - \cos(\omega) = 0$ $(\omega = 0, \pm 2\pi, \ldots)$, we find that $\operatorname{Re} \hat{\mu}$ is not strictly positive everywhere, i.e. μ is not strictly positive definite.

The result one gets by combining Theorem 1 and 2(i) is contained in a paper by Londen [4], which together with [7] has been the main source of inspiration for this study. In [4] Londen deduces an inequality of type (3) for the kernel in Theorem 2(i), and employs it to greatly simplify his original proof of [2, Theorem 1]. Recently Staffans [10] has given yet another proof of (a slightly weakened version of) [2, Theorem 1]. The argument in [10] is, however, more complicated than the one in [4] and than the one presented here.

Applying Theorems 1 and 2(ii) one can add the conclusion $x' - f \in L^2(R^+)$

to [6, Theorem 1(ii)] and get an alternative proof of [6, line (1.6)] in the special case when $a \in L^1(R^+) \cap BV(R^+)$.

Combining Theorems 1 and 2(iii) we get

COROLLARY 1. *Let $g \in C(R)$, $f \in L^1(R^+)$, $a \in L^1(R^+)$, and suppose that a, $-a'$ are nonnegative and convex. Then every bounded solution x of*

$$x'(t) + \int_{[0,t]} g(x(t-s))a(s)\,ds = f(t) \qquad (t \in R^+)$$

satisfies $x' - f \in L^2(R^+)$. If moreover $a \not\equiv 0$, then $g(x(t)) \to 0$ $(t \to \infty)$.

The second conclusion of Corollary 1 is well known, and is valid under weaker assumptions on a than those given here (see e.g. [6, Theorem 1(ii) and Corollary 2.2]). However, the first conclusion is new.

2. **Proof of Lemma 1.** The argument presented below is quite similar to the one in [7, §4], and therefore we omit a detailed motivation of each step (the extension to R of the function a in [7, Lemma 2] is replaced by the extension used in [8, Lemma 1.1]).

Define

$$\varphi_T = \chi_{[0,T]}\varphi; \quad \hat{\varphi}_T(\omega) = \int_{[0,T]} e^{-i\omega t}\varphi(t)\,dt.$$

Then one has

$$\begin{aligned}\alpha \int_{[0,T]} \varphi(t) \int_{[0,t]} \varphi(t-s)\,d\mu(s)\,dt &= \frac{\alpha}{2\pi}\int_R |\hat{\varphi}_T(\omega)|^2 \operatorname{Re}\hat{\mu}(\omega)\,d\omega \\ &\geqslant \frac{1}{2\pi}\int_R |\hat{\varphi}_T(\omega)\hat{\mu}(\omega)|^2\,d\omega = \int_R \left|\int_{[t-T,t]\cap R^+} \varphi(t-s)\,d\mu(s)\right|^2 dt \\ &\geqslant \int_{[0,T]} \left|\int_{[0,t]} \varphi(t-s)\,d\mu(s)\right|^2 dt,\end{aligned}$$

where the first inequality follows from (2), and the other steps use only elementary properties of Fourier transforms.

3. **Proof of Theorem 1.** It is well known (argue as in [7, §2]) that the assumption of Theorem 1 yields

$$\sup_{T>0} \int_{[0,T]} g(x(t)) \int_{[0,t]} g(x(t-s))\,d\mu(s)\,dt < \infty.$$

Hence by Lemma 1,

$$\int_{R^+} \left|\int_{[0,t]} g(x(t-s))\,d\mu(s)\right|^2 dt < \infty,$$

which together with (1) gives $x' - f \in L^2(R^+)$ and verifies (i).

To prove part (ii) one applies [10, Theorem 4.2] (note that it follows from part (i) that every $y \in \Gamma(x)$ is a constant, which is required in [10, Theorem 4.2]). Part (iii) is demonstrated using a minor modification of the arguments in the second paragraph of the proof of [10, Theorem 4.1].

4. **Proof of Theorem 2(i).** By Hölder's inequality,

$$|\mathrm{Im}\ \hat{\mu}(\omega)|^2 = \left|\int_{(0,\infty)} \sin(\omega t)\, d\mu(t)\right|^2$$
$$\leqslant -[b(0) - b(\infty)] \int_{(0,\infty)} \sin^2(\omega t)\, d\mu(t).$$

Hence

$$2[b(0) - b(\infty)]\mathrm{Re}\ \hat{\mu}(\omega) - |\mathrm{Im}\ \hat{\mu}(\omega)|^2$$
$$\geqslant 2[b(0) - b(\infty)]\left\{b(0) + \int_{(0,\infty)} [\cos(\omega t) + \tfrac{1}{2}\sin^2(\omega t)]\, d\mu(t)\right\}$$
$$= 2[b(0) - b(\infty)]\left\{b(0) + \int_{(0,\infty)} [1 - \tfrac{1}{2}(1 - \cos(\omega t))^2]\, d\mu(t)\right\}$$
$$\geqslant 2[b(0) - b(\infty)]\left\{b(0) + \int_{(0,\infty)} d\mu(t)\right\}$$
$$= 2[b(0) - b(\infty)]b(\infty) \geqslant 0.$$

Combining this with the trivial estimate Re $\hat{\mu}(\omega) \leqslant 2b(0) - b(\infty)$ one obtains (2) with $\alpha = 4b(0) - 3b(\infty)$.

5. **Proof of Theorem 2(ii).** The fact that $a \in BV(R^+)$ gives

$$|\hat{\mu}(\omega)| = \left|\int_{R^+} e^{-i\omega t} a(t)\, dt\right| = \frac{1}{\omega}\left|\int_{R^+} (1 - e^{-i\omega t})\, da(t)\right|$$
$$\leqslant 2A/\omega \qquad (\omega \in R, \omega \neq 0),$$

where A is the total variation of a on R^+. This together with $|\hat{\mu}(\omega)| \leqslant \int_{R^+} |a(t)|\, dt$ yields the existence of a constant γ such that

$$|\hat{\mu}(\omega)|^2 \leqslant \gamma(1 + \omega^2)^{-1} \qquad (\omega \in R).$$

The strong positive definiteness of a then implies (2) with $\alpha = \gamma/\varepsilon$.

6. **Proof of Theorem 2(iii).** We first notice that one can use the monotonicity of a together with $a \in L^1(R^+)$ to get

$$ta(t) \leqslant 2\int_{[t/2,t]} a(s)\, ds = O(1) \qquad (t \to 0+, t \to \infty),$$

and then one can show inductively

$$|t^{k+1} a^{(k)}(t)| \leqslant 2t^k \int_{[t/2,t]} |a^{(k)}(s)|\, ds$$
$$\leqslant 2t^k |a^{(k-1)}(t/2)| = O(1) \qquad (t \to 0+, t \to \infty)$$

for $k = 1, 2$. These estimates justify the integrations by parts that are performed below.

We integrate by parts twice to get

$$\operatorname{Im} \hat{\mu}(\omega) = \frac{1}{\omega} \int_{R^+} t^{-2}\left(t - \frac{1}{\omega} \sin(\omega t)\right) t^2 a''(t)\, dt \qquad (\omega \neq 0).$$

Hence by Hölder's inequality and the fact that

$$\int_{R^+} t^2 a''(t)\, dt = 2 \int_{R^+} a(t)\, dt \stackrel{\text{def}}{=} 2A,$$

one has

$$|\operatorname{Im} \hat{\mu}(\omega)|^2 \leqslant 2A\omega^{-2} \int_{R^+} t^{-2}\left(t - \frac{1}{\omega} \sin(\omega t)\right)^2 a''(t)\, dt \qquad (\omega \neq 0).$$

One more integration by parts together with a change of variable yields

$$(4) \qquad |\operatorname{Im} \hat{\mu}(\omega)|^2 \leqslant -2A\omega^{-3} \int_{(0,\infty)} h(\omega t)\, da''(t) \qquad (\omega \neq 0),$$

where $h(t) = \int_{[0,t]} (1 - s^{-1} \sin(s))^2\, ds$.

On the other hand, one can integrate Re $\hat{\mu}$ by parts three times to get

$$(5) \qquad \operatorname{Re} \hat{\mu}(\omega) = -\omega^{-3} \int_{(0,\infty)} (\omega t - \sin(\omega t))\, da''(t) \qquad (\omega \neq 0).$$

We claim that

$$(6) \qquad h(t) \leqslant 2(t - \sin(t)) \qquad (t \in R^+).$$

Assume this for the moment. Then clearly (4) and (5) together with the convexity of $-a'$ imply $|\operatorname{Im} \hat{\mu}(\omega)|^2 \leqslant 4A \operatorname{Re} \hat{\mu}(\omega)$ $(\omega \neq 0)$. The same inequality is trivially true for $\omega = 0$. Combining this with the fact that $|\operatorname{Re} \hat{\mu}(\omega)| \leqslant A$ one gets (2) with $\alpha = 5A$. Thus it only remains to prove (5).

Using the power series expansion of $\sin(t)$ one can easily check that $h(t) \leqslant t^5/180$, $t - \sin(t) \geqslant t^3/6 - t^5/120$ $(t \in R^+)$. Hence, in particular, $h(t) \leqslant 2(t - \sin(t))$ $(t \leqslant 2)$. For the remaining values of t, i.e. for $t > 2$, we estimate

$$\begin{aligned} h(t) &\leqslant h(2) + (t-2) \sup_{s \in R^+} (1 - s^{-1} \sin(s))^2 \\ &< 32/180 + 2(t-2) < 2(t-1) \leqslant 2(t - \sin(t)). \end{aligned}$$

This yields (6), and completes the proof of Theorem 2(iii).

7. **A final comment**. We have throughout assumed that the kernel μ has a finite total variation (or that the function a in Theorem 2(ii)–(iii) is integrable). This condition can be weakened somewhat, but there is a built-in restriction in (2) which limits the class of kernels that can be treated with the method presented above. Note that (2) implies $|\hat{\mu}(\omega)| \leqslant \alpha$ $(\omega \in R)$, i.e. the Fourier transform of the kernel must be a bounded function. This necessary condition is in fact also sufficient, i.e. Lemma 1 and Theorem 1 are true for any positive definite measure μ whose Fourier transform is a bounded function satisfying (2).

NOTES ADDED IN PROOF. 1. One version of Lemma 1 is contained in the proof of Theorem 1 in V. Barbu, *Sur une équation intégrale non-linéaire*, An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. **10** (1964), 61–65. There it is used to prove the existence of a square integrable solution of a undifferentiated, nonlinear Volterra equation.

2. The results of [6] are summarized in J. A. Nohel and D. F. Shea, *On the global behavior of a nonlinear Volterra equation*, International Conference on Differential Equations, H. A. Antosiewicz, ed., Academic Press, New York, 1975.

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