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A KOROVKIN TYPE THEOREM ON WEAK CONVERGENCE

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ABSTRACT. A weak convergence theorem is proved for a sequence of positive contractions on $L_1(0, 1)$.

In this note we shall prove the following result.

THEOREM. *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of positive and contractive linear operators on $L_1(0, 1)$, satisfying*

$$T_n 1 \xrightarrow{w} 1, \quad T_n x \xrightarrow{w} x. \tag{1}$$

Then

$$T_n f \xrightarrow{w} f \tag{2}$$

for every $f \in L_1(0, 1)$.

For the proof we shall need a classical result.

LEMMA [1, COROLLARY IV. 8.11]. *A subset S of $L_1(0, 1)$ is weakly sequentially compact if and only if it is bounded and*

$$\lim_{\mu E \to 0} \int_E f = 0$$

uniformly for $f \in S$.

PROOF OF THE THEOREM. Let r, $0 \leqslant r \leqslant 1$, be fixed. Since $|x - r| \leqslant 1$ on $(0, 1)$, we have by the positivity of the operators for any measurable $E \subset (0, 1)$

$$\int_E T_n(|x - r|) \leqslant \int_E T_n 1.$$

By the Lemma, the right-hand integral tends to zero, uniformly for $n = 1, 2, \ldots$, as $\mu E \to 0$. Hence, again by the Lemma, $\{T_n(|x - r|)\}_{n=1}^{\infty}$ is weakly sequentially compact. It follows that there exists a subsequence $T_{n_j}(|x - r|)$ which converges weakly to some $g \in L_1(0, 1)$. Now let E be any measurable subset of $(0, 1)$. We have on account of (1) and the positivity of the operators

$$\int (x - r) \cdot \chi_E \leftarrow \int T_{n_j}(x - r) \cdot \chi_E \leqslant \int T_{n_j}(|x - r|) \cdot \chi_E \to \int g \cdot \chi_E.$$

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Thus $x - r \leqslant g$ a.e. Similarly $r - x \leqslant g$ a.e. Therefore $|x - r| \leqslant g$ a.e. On the other hand, it follows from the contractive nature of the operators that $\|g\| \leqslant \underline{\lim}\, \|T_{n_j}(|x - r|)\| \leqslant \|x - r\|$. The above imply that $g = |x - r|$ a.e., in other words

$$T_{n_j}(|x - r|) \xrightarrow{w} |x - r|.$$

By Cantor's diagonal argument, there exists a subsequence, say $\{l_j\}$, of $\{n_j\}$ such that

$$T_{l_j}(|x - r|) \xrightarrow{w} |x - r| \tag{3}$$

for every *rational* r.

The linear span of the functions 1, x, $\{|x - r|: r \text{ rational}\}$ is the set of polygonal lines with rational vertices; hence it is dense in $L_1(0, 1)$. From (1) and (3) it follows, therefore, that

$$T_{l_j} f \xrightarrow{w} f \tag{4}$$

for every $f \in L_1(0, 1)$.

Since from every subsequence of $\{T_n\}$ we can select a further subsequence satisfying (4), (2) must be true for the sequence $\{T_n\}$ itself.

The following result of Wulbert [2] is an immediate consequence of the Theorem.

COROLLARY. *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of positive and contractive linear operators on $L_1(0, 1)$ satisfying*

$$T_n 1 \xrightarrow{s} 1, \quad T_n x \xrightarrow{w} x. \tag{7}$$

Then $T_n f \xrightarrow{s} f$ for every $f \in L_1(0, 1)$.

PROOF. Let E be any measurable subset of $(0, 1)$ and $\tilde{E}$ its complement. Since $\chi_E = 1 - \chi_{\tilde{E}}$, we have on $(0, 1)$

$$T_n \chi_E - \chi_E = (T_n 1 - 1) \cdot \chi_E - (T_n \chi_{\tilde{E}}) \cdot \chi_E + (T_n \chi_E) \cdot \chi_{\tilde{E}}.$$

Hence

$$\|T_n \chi_E - \chi_E\| \leqslant \|T_n 1 - 1\| + \int (T_n \chi_{\tilde{E}}) \cdot \chi_E + \int (T_n \chi_E) \cdot \chi_{\tilde{E}}.$$

The first quantity on the right side tends to zero on account of (7), the second and third tend to zero by our Theorem. Hence $T_n \chi_E \xrightarrow{s} \chi_E$ for every measurable E. The conclusion now follows, since $\{\chi_E\}$ are dense in $L_1(0, 1)$.

REMARKS. (i) The assumption $\|T_n\| \leqslant 1$ cannot be removed. For, if

$$(Tf)(x) = \begin{cases} \dfrac{3}{2} \displaystyle\int_{x-1/3}^{x+1/3} f, & \dfrac{1}{3} < x < \dfrac{2}{3}, \\ f(x), & \text{otherwise.} \end{cases}$$

Then $T1 \equiv 1$, $Tx \equiv x$, but $T\chi_E = \frac{1}{2}\chi_E$ for $E = [\frac{1}{3}, \frac{2}{3}]$.

(ii) The conclusion $T_n f \xrightarrow{w} f$ cannot be replaced by $T_n f \xrightarrow{s} f$. For, if

$$(T_n f)(x) = \begin{cases} 2n \int_{k/n}^{(k+1)/n} f, & \dfrac{k}{n} < x < \dfrac{2k+1}{2n};\ 0 \leqslant k < n, \\ 0, & \text{otherwise,} \end{cases}$$

then $T_n f \xrightarrow{w} f$ for every $f \in L_1(0, 1)$, but $\|T_n 1 - 1\| = 1$. This example is due to J. B. Gamlen.

References

1. N. Dunford and J. T. Schwartz, *Linear operators.* I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR **22** #8302.

2. D. E. Wulbert, *Convergence of operators and Korovkin's theorem*, J. Approximation Theory **1** (1968), 381–390. MR **38** #3679.

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