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CYCLICALLY MONOTONE LINEAR OPERATORS[1]

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ABSTRACT. A linear operator on a complex Hilbert space $\mathcal{H}$ is called n-cyclically monotone if for each sequence $x_0, x_1, \ldots, x_{n-1}, x_n = x_0$ of n elements in $\mathcal{H}$, $\Sigma_{j=0}^{n-1}\mathrm{Re}(Tx_j - x_{j+1}) \geqslant 0$. We show that T is n-cyclically monotone if and only if $|\mathrm{Arg}(Tx, x)| \leqslant \pi/n$, $\forall x \in \mathcal{H}$. If T_m and T_n are m- and n-cyclically monotone operators, then the spectrum of the product $T_m T_n$ lies in the sector $\{z \in \mathbf{C}\colon |\mathrm{Arg}\, z| \leqslant \pi/m + \pi/n\}$.

1. **Introduction.** Let H denote a real Hilbert space with inner product $(\cdot,\cdot)$. The following is a simplified version of [1, Theorem 3]: Let f and f_1 be two continuous (not necessarily linear) functions on H, mapping bounded subsets into bounded subsets, such that (i) f is monotone, i.e., $(f(x) - f(y), x - y) \geqslant 0$, $\forall x, y \in H$, (ii) f_1 is tricyclically monotone, i.e., $(f_1(x), x - y) + (f_1(y), y - z) + (f_1(z), z - x) \geqslant 0$, $\forall x, y, z \in H$. Then $I + ff_1$ is a homeomorphism.

This paper is motivated by the theorem above and we shall restrict our discussion to the elements in $\mathfrak{B}(\mathcal{H})$, the set of bounded linear operators on a complex Hilbert space $\mathcal{H}$. An operator $T \in \mathfrak{B}(\mathcal{H})$ is called n-cyclically monotone, n an integer greater than one, if for each sequence $x_0, x_1, x_2, \ldots, x_{n-1}, x_n = x_0$ of n points in $\mathcal{H}$, $\Sigma_{j=0}^{n-1}\mathrm{Re}(Tx_j, x_j - x_{j+1}) \geqslant 0$. A 2-cyclically monotone operator will be called accretive [5, p. 279]. The concept of the cyclically monotone operators was first introduced by R. T. Rockafellar [6]. According to [6], an n-cyclically monotone operator should be called monotone of degree $(n - 1)$; however, we justify our definition with the following theorem: T is n-cyclically monotone if and only if

$$|\mathrm{Arg}(Tx, x)| \leqslant \pi/n, \quad \forall x \in \mathcal{H}.$$

In the last section of this paper we show that if T is accretive and T_1 is 3-cyclically monotone, then for each λ in the spectrum of TT_1, $|\mathrm{Arg}\,\lambda| \leqslant \pi/2 + \pi/3$; consequently $I + TT_1$ is invertible.

2. **Notation and preliminaries.** Let $\mathbf{C}$, $\mathbf{R}$ and $\mathbf{R}^+$ denote the set of complex, real and nonnegative real numbers, respectively. Let $\Omega, \Omega_1 \subset \mathbf{C}$, $\Omega \cdot \Omega_1 = \{zz_1\colon z \in \Omega, z_1 \in \Omega_1\}$; $\mathrm{Cl}(\Omega)$ denotes the closure and $\mathrm{Co}(\Omega)$ the convex hull of Ω. For $\alpha, \beta \in \mathbf{R}$, $0 \leqslant \beta - \alpha \leqslant 2\pi$, $\Sigma(\alpha, \beta)$ denotes the closed sector $\{z \in \mathbf{C}\colon \alpha \leqslant \arg z \leqslant \beta\}$. For $\alpha \in \mathbf{R}$, $0 \leqslant \alpha \leqslant \pi$, $\Sigma(\alpha)$ denotes the symmetric sector $\Sigma(-\alpha, \alpha)$.

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For $T \in \mathfrak{B}(\mathcal{H})$, $\operatorname{Re} T = (T + T^*)/2$ and $\operatorname{Im} T = (T - T^*)/2i$; $\sigma(T)$ denotes the spectrum and $W(T)$ the numerical range of T, $W(T) = \{(Tx, x): \|x\| = 1\}$. T is called nonnegative if $W(T) \subset [0, \infty)$. We define $A(T) = \mathrm{Cl}(\{(Tx, x)\})$. Since $A(T) = \mathbf{R}^+ \cdot \mathrm{Cl}(W(T))$ and the numerical range of an operator is convex, either $A(T) = \mathbf{C}$ or $A(T) = \Sigma(\alpha, \beta)$ with $\beta - \alpha \leqslant \pi$. If $\mathcal{H}$ is finite dimensional and $0 \in W(T)$, then $A(T)$ coincides with the angular field introduced in [10].

LEMMA 1. *Let $T, S \in \mathfrak{B}(\mathcal{H})$. If S is invertible, then $A(T) = A(S^*TS)$.*

LEMMA 2 [5, VI-§1.2]. *Let $T \in \mathfrak{B}(\mathcal{H})$ and $\alpha \in [0, \pi/2)$; then the following three statements are equivalent*:

(1) $A(T) \subset \Sigma(\alpha)$;

(2) $|(\operatorname{Im} Tx, x)| \leqslant \tan(\alpha)(\operatorname{Re} Tx, x)$, $\forall x \in \mathcal{H}$;

(3) $|(\operatorname{Im} Tx, y)| \leqslant \tan(\alpha)[(\operatorname{Re} Tx, x)(\operatorname{Re} Ty, y)]^{1/2}$, $\forall x, y \in \mathcal{H}$.

Furthermore, each of these conditions implies

(4) $\|Tx\|^2 \leqslant (1 + \tan(\alpha))^2 \|\operatorname{Re} T\|(\operatorname{Re} Tx, x)$, $\forall x \in \mathcal{H}$.

Let $S(n)$ denote the n-by-n backward-shift matrix, i.e., $S(n) = (\delta_{i+1,j})_{n \times n}$. Let $R(n) = (I - S(n))^{-1}$, then $R(n)$ is the n-by-n matrix with 1's on and above the diagonal and 0's below the diagonal.

LEMMA 3. $A(R(n)) = \Sigma(\pi/2 - \pi/(n + 1))$.

PROOF. Since $R(n)$ is a real matrix, $A(R(n)) = A(I - S(n))$. The result follows if we show that $W(S(n))$ is a disc centered at 0 with radius $\cos(\pi/n + 1)$. It is easy to see that $W(S(n))$ is a disc centered at 0. The numerical radius of $S(n)$ is the spectral radius of $\operatorname{Re} S(n)$. Put $U_m(\lambda) = \det(2\lambda - S(m) - S(m)^*)$, $m = 2, 3, \ldots$. If we define $U_0(\lambda) = 1$ and $U_1(\lambda) = 2\lambda$, then $U_m(\lambda) = 2\lambda U_{m-1}(\lambda) - U_{m-2}(\lambda)$, $m = 2, 3, \ldots$. We notice that $U_m(\lambda)$ satisfies the recurrence relations and initial conditions of the Chebyshev polynomial of the second kind [9, p. 128]. Thus

$$U_m(\lambda) = \sin((m + 1)\operatorname{arc}\cos(\lambda))/\sin(\operatorname{arc}\cos(\lambda)).$$

Consequently the numerical radius of $S(n)$ is $\cos(\pi/n + 1)$. □

PROPOSITION [2]. *Let $S, T \in \mathfrak{B}(\mathcal{H})$ and let $S \otimes T$ denote the tensor product acting on the product space $\mathcal{H} \otimes \mathcal{H}$. Then $\sigma(S \otimes T) = \sigma(S) \cdot \sigma(T)$.*

COROLLARY. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. For $T_j \in \mathfrak{B}(\mathcal{H}_j)$, $j = 1, 2$, $\sigma(T_1 \otimes T_2) = \sigma(T_1) \cdot \sigma(T_2)$.*

LEMMA 4 [8]. *Let $T_j \in \mathfrak{B}(\mathcal{H}_j)$ be a normal operator, $j = 1, 2$. Then* $\mathrm{Cl}(W(T_1 \otimes T_2)) = \mathrm{Cl}(\mathrm{Co}(W(T_1) \cdot W(T_2)))$.

PROOF. $T_1 \otimes T_2$ is also normal.

$$\begin{aligned} \text{L.H.S.} &= \mathrm{Co}(\sigma(T_1 \otimes T_2)) \\ &= \mathrm{Co}(\sigma(T_1) \cdot \sigma(T_2)) \quad \text{by Corollary} \\ &= \mathrm{Co}(\mathrm{Co}(\sigma(T_1)) \cdot \mathrm{Co}(\sigma(T_2))) \\ &= \mathrm{Co}(\mathrm{Cl}(W(T_1)) \cdot \mathrm{Cl}(W(T_2))) = \text{R.H.S.} \quad \square \end{aligned}$$

3. Characterizations of cyclically monotone linear operators.

THEOREM 1. *Let $T \in \mathfrak{B}(\mathcal{H})$. The following statements are equivalent.*
(1) *T is n-cyclically monotone.*
(2) *For every sequence $y_1, \ldots, y_{n-1}$ of $(n-1)$ points in $\mathcal{H}$,*

$$\sum_{j=1}^{n-1} \operatorname{Re}\left(Ty_j, \sum_{k=1}^{j} y_k\right) \geqslant 0.$$

(3) *The operator $R(n-1) \otimes T$ on $\mathbf{C}^{n-1} \otimes \mathcal{H}$ is accretive.*
(4) $A(T) \subset \Sigma(\pi/n)$.

PROOF. (1) $\Leftrightarrow$ (2).

$$\sum_{j=0}^{n-1} (Tx_j, x_j - x_{j+1}) = \sum_{j=1}^{n-1} (Tx_j - Tx_{j-1}, x_j - x_0)$$
$$= \sum_{j=1}^{n-1} \left(Ty_j, \sum_{k=1}^{j} y_k\right), \quad \text{where } y_k = x_k - x_{k-1}.$$

(2) $\Leftrightarrow$ (3).

$$\sum_{j=1}^{n-1} \left(Ty_j, \sum_{k=1}^{j} y_k\right) = \sum_{j=1}^{n-1} \left(y_j, \sum_{k=1}^{j} T^*y_k\right)$$

$$= \left(\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix}, \begin{bmatrix} T^*y_1 \\ T^*y_1 + T^*y_2 \\ \vdots \\ T^*y_1 + T^*y_2 + \cdots + T^*y_{n-1} \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix}, \begin{bmatrix} T^* & & & O \\ T^* & T^* & & \\ \vdots & & \ddots & \\ T^* & T^* & \cdots & T^* \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} \right)$$

$$= \left(R(n-1) \otimes T \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} \right).$$

(3) $\Leftrightarrow$ (4) is an immediate consequence of Lemma 3 and

THEOREM 2. *Let $T_j \in \mathfrak{B}(\mathcal{H}_j)$ and $A(T_j) = \Sigma(\alpha_j, \beta_j), j = 1, 2$. Suppose either* (i) *$A(T_1 \otimes T_2) \neq \mathbf{C}$, or* (ii) *$(\beta_1 - \alpha_1) + (\beta_2 - \alpha_2) \leqslant \pi$; then $A(T_1 \otimes T_2) = \Sigma(\alpha_1 + \alpha_2, \beta_1 + \beta_2)$.*

PROOF. Since it is always true that $W(T_1 \otimes T_2) \supset W(T_1) \cdot W(T_2)$,

$$A(T_1 \otimes T_2) \supset \Sigma(\alpha_1 + \alpha_2, \beta_1 + \beta_2).$$

Consequently, assumption (i) implies assumption (ii).

To show that $A(T_1 \otimes T_2) \subset \Sigma(\alpha_1 + \alpha_2, \beta_1 + \beta_2)$, we need only to establish the case $\alpha_j = -\beta_j$, i.e., $A(T_j) = \Sigma(\beta_j)$, $j = 1, 2$. Write $T_j = \operatorname{Re} T_j + \operatorname{Im} T_j$, $j = 1, 2$. Since $\operatorname{Re} T_j$ is nonnegative, it has a nonnegative square root Q_j [5, Theorem V.3.35(iv)]. Furthermore, if we assume that $\operatorname{Re} T_j$ is invertible, then $T_j = Q_j N_j Q_j$, where N_j is the normal operator $I + iQ_j^{-1}(\operatorname{Im} T_j)Q_j^{-1}$, $j = 1, 2$. Thus $T_1 \otimes T_2 = (Q_1 \otimes Q_2)(N_1 \otimes N_2)(Q_1 \otimes Q_2)$.

$$\begin{aligned} A(T_1 \otimes T_2) &= A(N_1 \otimes N_2) \quad \text{by Lemma 1} \\ &= \mathbf{R}^+ \cdot \operatorname{Cl}(W(N_1 \otimes N_2)) \\ &= \mathbf{R}^+ \cdot \operatorname{Cl}(\operatorname{Co}(W(N_1)W(N_2))) \quad \text{by Lemma 4} \\ &= \Sigma(\beta_1 + \beta_2) \quad \text{by assumption (ii).} \end{aligned}$$

Thus the theorem is proved if both $\operatorname{Re} T_1$ and $\operatorname{Re} T_2$ are invertible. In general, we have $A((T_1 + \varepsilon) \otimes (T_2 + \varepsilon)) \subset \Sigma(\beta_1 + \beta_2)$ for each $\varepsilon > 0$. $\operatorname{Cl}(W(\cdot))$ is continuous with respect to the uniform operator topology [3, Problem 175]; we let ε tend to 0 and obtain $A(T_1 \otimes T_2) \subset \Sigma(\beta_1 + \beta_2)$. □

Theorem 1 answers the conjecture raised in [6, p. 500]. The following corollary is a complex linear operator version of [6, Theorem 1] and [7, Theorem 24.8].

COROLLARY 1. *For $T \in \mathcal{B}(\mathcal{H})$, T is nonnegative if and only if T is n-cyclically monotone, $n = 2, 3 \ldots$.*

REMARKS. Since the concept of an n-cyclically monotone operator is in essence a finite dimensional one, Theorem 1 can be rephrased for the cases of unbounded operators or sectorial sesquilinear forms [5, §VI-1.2]. An n-cyclically monotone linear operator, if defined on the whole Hilbert space, is necessarily bounded [5, Theorem V.3.4].

4. **Spectra of products.** In this section we study the spectrum location of the product of two operators.

THEOREM 3 [10, THEOREM 2], [11, THEOREM 1]. *Let $S, T \in \mathcal{B}(\mathcal{H})$. If $0 \notin \operatorname{Cl}(W(T))$, then $\{\sigma(ST) \cup \sigma(TS)\} \subset \operatorname{Cl}(W(S))/\operatorname{Cl}(W(T^{-1}))$.*

PROOF. We note that the nonzero elements of $\sigma(ST)$ and $\sigma(TS)$ are the same [3, Problem 61], and $0 \in \operatorname{Cl}(W(T))$ if and only if $0 \in \operatorname{Cl}(W(T^{-1}))$. If $0 \in \sigma(ST - \lambda)$, then

$$\begin{aligned} 0 \in \sigma(S - \lambda T^{-1}) &\subset \operatorname{Cl}\big(W(S - \lambda T^{-1})\big) \\ &\subset \operatorname{Cl}(W(S)) - \lambda \cdot \operatorname{Cl}\big(W(T^{-1})\big). \quad \square \end{aligned}$$

Thus for an m-cyclically monotone operator S and an n-cyclically monotone operator T, $\{\sigma(ST) \cup \sigma(TS)\} \subset \Sigma(\pi/m + \pi/n)$ if $0 \notin \operatorname{Cl}(W(S))$ or

$0 \notin \mathrm{Cl}(W(T))$. We conclude this paper by showing that the last assumption is not necessary.

THEOREM 4 (cf. [4]). *Let* $S, T \in \mathfrak{B}(\mathcal{H})$ *with* S *accretive and* T *satisfying the condition*:

($*$) *There exists a constant* $d > 0$ *such that* $\mathrm{Re}(Tx, x) \geqslant d\|Tx\|^2$, $\forall x \in \mathcal{H}$. *Then* $(-\infty, 0) \cap \{\sigma(ST) \cup \sigma(TS)\} = \varnothing$.

PROOF. Let λ be a point in the approximate point spectrum of ST, i.e., there exists a sequence $\{x_n\}$ of unit vectors such that $\|(\lambda - ST)x_n\| \to 0$. Since $(\lambda x_n, Tx_n) - (STx_n, Tx_n) \to 0$ and S is accretive, $\liminf \mathrm{Re}(\lambda x_n, Tx_n) = \liminf \mathrm{Re}(STx_n, Tx_n) \geqslant 0$. If we assume $\lambda < 0$, then $\limsup \mathrm{Re}(x_n, Tx_n) \leqslant 0$. By ($*$), $\mathrm{Re}(x_n, Tx_n) \geqslant d\|Tx_n\|^2$; consequently, $\|Tx_n\| \to 0$ and this contradicts $\lambda \neq 0$. Thus the approximate point spectrum of ST has no negative numbers, and therefore the boundary of $\sigma(ST)$ has no negative numbers [3, Problem 63]. Hence $(-\infty, 0) \cap \sigma(ST) = \varnothing$. □

For $T \in \mathfrak{B}(\mathcal{H})$, if $A(T) \subset \Sigma(\alpha)$ with $\alpha < \pi/2$, then T satisfies ($*$) by the last part of Lemma 2. However, the converse does not hold; the example in [4, p. 309] is also valid for the complex case.

THEOREM 5. *Let* $T_j \in \mathfrak{B}(\mathcal{H})$ *with* $A(T_j) = \Sigma(\alpha_j, \beta_j)$, $j = 1, 2$. *Suppose* $(\beta_1 - \alpha_1) + (\beta_2 - \alpha_2) < 2\pi$; *then*

$$\{\sigma(T_1T_2) \cup \sigma(T_2T_1)\} \subset \Sigma(\alpha_1 + \alpha_2, \beta_1 + \beta_2).$$

PROOF. Consider the operators $e^{i\theta_j}T_j$, $j = 1, 2$; vary the real numbers θ_1 and θ_2 and apply Theorem 4. □

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REFERENCES

1. H. Brezis and F. E. Browder, *Maximal monotone operators in nonreflexive Banach spaces and nonlinear integral equations of Hammerstein type*, Bull. Amer. Math. Soc. **81** (1975), 82–88.

2. A. Brown and C. Pearcy, *Spectra of tensor products of operators*, Proc. Amer. Math. Soc. **17** (1966), 162–166. MR **32** #6218.

3. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N. J., 1967. MR **34** #8178.

4. P. Hess, *On nonlinear equations of Hammerstein type in Banach spaces*, Proc. Amer. Math. Soc. **30** (1971), 308–312. MR **43** #7980.

5. T. Kato, *Perturbation theory for linear operators*, Die Grundlehren der math. Wissenschaften, Band 132, Springer-Verlag, New York, 1966. MR **34** #3324.

6. R. T. Rockafellar, *Characterization of subdifferentials of convex functions*, Pacific J. Math. **17** (1966), 497–510. MR **33** #1769.

7. ———, *Convex analysis*, Princeton Math. Ser., no. 28, Princeton Univ. Press, Princeton, N. J., 1970. MR **43** #445.

8. T. Saitô, *Numerical ranges of tensor products of operators*, Tôhoku Math. J. (2) **19** (1967), 98–100. MR **37** #4654.

9. J. Todd (Editor), *Survey of numerical analysis*, McGraw-Hill, New York, 1962. MR **24** #B1271.

10. H. Wielandt, *On the eigenvalues of $A + B$ and AB*, J. Res. Nat. Bur. Standards Sect. B **77B** (1973), 61–63. MR **49** #318.

11. J. P. Williams, *Spectra of products and numerical ranges*, J. Math. Anal. Appl. **17** (1967), 214–220. MR **34** #3341.

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