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PRODUCTS OF HERMITIAN OPERATORS

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ABSTRACT. Let A and B be selfadjoint operators on a Hilbert space. It is shown that AB and BA are not necessarily similar if their null spaces have equal dimension. If A and B are assumed to be Fredholm, then similarity can be established if additional conditions are satisfied.

1. **Introduction.** Let A and B be bounded Hermitian operators on a Hilbert space $\mathcal{H}$. It is known that AB is similar to its adjoint BA when $\mathcal{H}$ is finite dimensional, and that this is not necessarily true when the dimension of $\mathcal{H}$ is infinite [2]. Radjavi and Williams have asked whether the condition $\dim N(AB) = \dim N(BA)$ (where $\dim N(X)$ is the dimension of the null space of X) is sufficient to guarantee the similarity of AB and BA. In this note we give an example to show that it is not sufficient. This example indicates that little can be said in general if either A or B has nonclosed range, so we have investigated the situation when both A and B are Fredholm (closed range and finite dimensional null space). We conjecture that AB and BA are similar in this case, and we have been able to prove it under several different additional assumptions. Our main result is that similarity holds if A and B are Fredholm and either one is positive.

2. **An example.** In this section we will present the promised example. We let $\mathcal{H}$ be $L_2[0, 1]$, and we define the following two elements of $\mathcal{H}$:

$$g(x) = \begin{cases} -1, & 0 \leqslant x \leqslant 1/2, \\ 1, & 1/2 < x \leqslant 1, \end{cases}$$

$$f(x) = \begin{cases} x, & 0 \leqslant x \leqslant 1/2, \\ 1 - x, & 1/2 \leqslant x \leqslant 1. \end{cases}$$

Let P be the orthogonal projection onto $\{\lambda f\}$, the one-dimensional space spanned by f, and define $A = 1 - P$. Finally, for $k(x) \in \mathcal{H}$ we define B via

$$(Bk)(x) = \int_{1-x}^{1} k(t)\,dt.$$

It is easily checked that B is Hermitian.

Since $Bg = f$, we have $N(AB) = \{\lambda g\}$, and clearly, $N(BA) = \{\lambda f\}$. Thus, both null spaces are one dimensional; we will show that AB and BA are not

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similar by showing $\dim N([AB]^2) = 1$ while $\dim N([BA]^2) = 2$. Notice that $(f, g) = 0$, which implies $Ag = g$, and therefore $ABAg = Af = 0$; hence $g \in N([BA]^2)$. If $\dim N([AB]^2) > 1$, then g belongs to the range of AB, and thus $g + \lambda f$ is in the range of B for some scalar λ. But this is impossible because the range of B consists of absolutely continuous functions. This completes the proof.

3. **Fredholm operators.** In this section, $\mathfrak{R}_X$ will denote the orthogonal projection onto the closure of the range of X, and N_X will be the projection onto the null space. If P is a projection, the symbol P will also be used to denote the subspace $P(\mathcal{H})$; it will be clear from the context which is meant. For the basic facts about Fredholm operators which will be used, the reader is referred to [1]. If S and T are similar, we will write $S \sim T$. Radjavi and Williams [2] have conjectured that every operator which is similar to its adjoint is a product of two Hermitians; if this is true, then whenever $AB \sim BA$, we should be able to find an invertible Hermitian H such that $HAB = BAH$. In all of the following results, the operator which establishes the similarity will be Hermitian. We begin with a simple extension of the finite-dimensional result.

Proposition 1. *If A and B are Hermitian and $\mathfrak{R}_B$ is finite, then $AB \sim BA$.*

Proof. Let $Q = \sup(\mathfrak{R}_B, \mathfrak{R}_{AB})$; Q is invariant for both B and AB and thus

$$ABQ = QABQ = QAQBQ = (QAQ)(QBQ).$$

Since Q is finite, there is an invertible Hermitian H mapping Q onto Q such that $HABQ = QBAH = BAH$. If we define H to be the identity on $1 - Q$, then it is easily seen that $BAH = HAB$, and thus $AB \sim BA$. □

The main interest of Proposition 1 is that it does not extend to the case when B is compact. In the above example, A is as simple as possible–a projection with one-dimensional null space–and B is compact, and similarity does not hold.

Lemma 1. *If E_n is n-dimensional Euclidean space, P, Q two subspaces of dimension m, then there exists an invertible Hermitian H on E_n such that H maps P onto Q.*

Proof. We note that the result is obvious if P and Q are orthogonal; further, we know that $PQ \sim QP$, and therefore $\dim N(PQ) = \dim N(QP)$. But $\dim N(PQ) = \dim(1 - Q) + \dim(Q \cap 1 - P)$ and likewise for QP. However, $\dim(1 - Q) = \dim(1 - P) = n - m$, and thus $\dim(Q \cap 1 - P) = \dim(P \cap 1 - Q)$. From our above remark, and since $Q \cap 1 - P$ is orthogonal to $P \cap 1 - Q$, we may assume that $P \cap 1 - Q = Q \cap 1 - P = 0$. But now $\dim \mathfrak{R}_{PQ} = m$, and therefore $\mathfrak{R}_{PQ} = P$, $\mathfrak{R}_{QP} = Q$. Now $HPQ = QPH$ for an invertible Hermitian H, and clearly H satisfies the requirement of the lemma.

Proposition 2. *If $N_B \cap \mathfrak{R}_{AB} = \{0\}$, where A and B are Hermitian and Fredholm, then $AB \sim BA$.*

PROOF. We will show that there exists a mapping T, defined on the finite dimensional space N_{BA}, which is invertible, Hermitian, and such that $H = \lambda B + TN_{BA}$ is invertible for some λ. Clearly, for any H of this form, $HAB = BAH$.

Since B is Fredholm and N_{BA} is compact, H is Fredholm for any choice of T and $\lambda \neq 0$; hence, to verify invertibility, it suffices to show $N_H = 0$. Let $x \in N_H$, and write $x = x_r + x_n$, where $x_r \in \mathfrak{R}_B$, $x_n \in N_B$. Since $N_B \cap \mathfrak{R}_{AB} = \{0\}$, $Bx \neq 0$, and thus $x_r \neq 0$. Now,

$$-\lambda Bx_r = TN_{BA}x_r + TN_{BA}x_n \tag{2.0}$$

implies $Bx_r \in N_{BA}$, or $x_r \in \mathfrak{R}_B \cap N_{BAB}$. But $N_{BAB} = N_{AB}$, so $x_r \in \mathfrak{R}_B \cap N_{AB} = N_{AB} - N_B$. Since AB is Fredholm of index 0, $\dim(N_{AB}) = \dim(N_{BA})$, and thus

$$\dim(\{Bx\} \mid x \in N_{AB} - N_B) \leqslant \dim(N_{BA}) - \dim(N_B). \tag{2.1}$$

Also note that $\dim(\mathfrak{R}_{N_{BA}N_B}) \leqslant \dim(N_B)$ implies

$$\dim(N_{BA} - \mathfrak{R}_{N_{BA}N_B}) \geqslant \dim(N_{BA}) - \dim(N_B). \tag{2.2}$$

From (2.1), (2.2), and Lemma 1, we can find an invertible Hermitian $V\colon N_{BA} \to N_{BA}$ such that $VBx \perp \mathfrak{R}_{N_{BA}N_B}$ for all $x \in N_{AB} - N_B$. Letting $T = V^{-1}$ in (2.0), we get

$$\lambda\|VBx_r\| = \|N_{BA}x_r + N_{BA}x_n\| \leqslant \|N_{BA}x_r\| \leqslant \|x_r\|.$$

However, B has closed range, which means that $\|By\| \geqslant \varepsilon\|y\|$ for all $y \in \mathfrak{R}_B$ and some $\varepsilon > 0$. Similarly, $\|Vy\| \geqslant \delta\|y\|$ for $y \in N_{BA}$ by the invertibility of V. Hence,

$$\lambda\varepsilon\delta\|x_r\| \leqslant \lambda\|VBx_r\| \leqslant \|x_r\|$$

which is clearly violated if $\lambda > 1/\varepsilon\delta$. Thus, H is invertible, and the proof is complete. □

COROLLARY 1. *If A and B are Hermitian, Fredholm, and $A \geqslant 0$, then $AB \sim BA$.*

PROOF. Let $A_0 = (A + N_A)^{-1/2}$, which exists because $A + N_A$ is invertible and positive. If $B_1 = A_0^{-1}BA_0^{-1}$, then

$$A_0ABA_0^{-1} = (A_0AA_0)(A_0^{-1}BA_0^{-1}) = A_0AB_1 = \mathfrak{R}_AB_1.$$

We may therefore assume that A is a projection. However, $N_{X*X} = N_X$ for all operators, so $N_{AB} = N_{BAAB} = N_{BAB}$ which says that $N_B \cap \mathfrak{R}_{AB} = \{0\}$, and thus Proposition 2 applies. □

The above corollary may also be deduced from Proposition 3, the proof of which requires the following lemma. This lemma was demonstrated by Radjavi and Williams [2] in the course of proving their Theorem 5, so the proof will be omitted.

LEMMA 2. *If P and Q are linearly independent invariant subspaces of X which* *span $\mathcal{H}$, then there exists an invertible operator R such that*

(a) $X_1 = R^{-1}XR$ *commutes with* P,

(b) $Rp = p$ *for* $p \in P$,

(c) $\sigma(X|_P) = \sigma(X_1|_P)$ *where* σ *is the spectrum and* $X|_P$ *is* X *considered as an operator from* P *into* P.

An operator X is said to have *finite descent n* if n is the smallest integer for which X^n and X^{n+1} have the same range. For a Fredholm operator, finite descent is equivalent to zero being an isolated point of the spectrum.

PROPOSITION 3. *If A and B are Hermitian and Fredholm, and if AB has finite descent n, then $AB \sim BA$.*

PROOF. Since AB has index 0, $N([AB]^n) = N([AB]^{n+1})$; let $N_0 = N([AB]^n)$, $R_0 = \mathfrak{R}([AB]^n)$. Theorem 5.41-G of [3] applies, and so $\{N_0, R_0\}$ satisfies the conditions of Lemma 2, with AB playing the role of X. Note that N_0 is finite, and that $\sigma(AB|_{N_0}) = 0$. Since $R^{-1}ABR = (R^{-1}AR^{-1*})(R^*BR)$ is also a product of Hermitians, Lemma 2 says (with $N_0 = P$) that we may assume that AB commutes with N_0; i.e., $AB = \left(\begin{smallmatrix} V & 0 \\ 0 & U \end{smallmatrix}\right)$ where $\mathcal{H} = N_0 \oplus N_0^{\perp}$. By (b) of Lemma 2, $N_U = 0$, and since U is Fredholm with index 0, U is invertible. We also have $V^n = 0$. Now $B(AB)^n$ is Hermitian, and $N_{B(AB)^n} = N_{(AB)^n} = N_0$, and thus $\mathfrak{R}_{B(AB)^n} = N_0^{\perp} = \mathfrak{R}_{(AB)^n}$. Therefore, $N_0^{\perp}$ is a reducing subspace for B on which B is invertible. Let $B_1 = B|_{N_0^{\perp}}$. Furthermore, by Theorem 1 of [2] and the fact that $\sigma(V) = 0$, we can find an invertible Hermitian H defined on N_0 such that $HV = V^*H$. Hence,

$$\begin{pmatrix} H & 0 \\ 0 & B_1 \end{pmatrix}\begin{pmatrix} V & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} V^* & 0 \\ 0 & U^* \end{pmatrix}\begin{pmatrix} H & 0 \\ 0 & B_1 \end{pmatrix}$$

and

$$\begin{pmatrix} H & 0 \\ 0 & B_1 \end{pmatrix}$$

is invertible. $\square$

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