PRODUCTS OF HERMITIAN OPERATORS

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ABSTRACT. Let $A$ and $B$ be selfadjoint operators on a Hilbert space. It is shown that $AB$ and $BA$ are not necessarily similar if their null spaces have equal dimension. If $A$ and $B$ are assumed to be Fredholm, then similarity can be established if additional conditions are satisfied.

1. Introduction. Let $A$ and $B$ be bounded Hermitian operators on a Hilbert space $\mathcal{H}$. It is known that $AB$ is similar to its adjoint $BA$ when $\mathcal{H}$ is finite dimensional, and that this is not necessarily true when the dimension of $\mathcal{H}$ is infinite [2]. Radjavi and Williams have asked whether the condition $\dim N(AB) = \dim N(BA)$ (where $\dim N(X)$ is the dimension of the null space of $X$) is sufficient to guarantee the similarity of $AB$ and $BA$. In this note we give an example to show that it is not sufficient. This example indicates that little can be said in general if either $A$ or $B$ has nonclosed range, so we have investigated the situation when both $A$ and $B$ are Fredholm (closed range and finite dimensional null space). We conjecture that $AB$ and $BA$ are similar in this case, and we have been able to prove it under several different additional assumptions. Our main result is that similarity holds if $A$ and $B$ are Fredholm and either one is positive.

2. An example. In this section we will present the promised example. We let $\mathcal{H}$ be $L^2[0,1]$, and we define the following two elements of $\mathcal{H}$:

\[
g(x) = \begin{cases} 
-1, & 0 \leq x < 1/2, \\
1, & 1/2 \leq x \leq 1,
\end{cases}
\]

\[
f(x) = \begin{cases} 
x, & 0 \leq x < 1/2, \\
1 - x, & 1/2 \leq x \leq 1.
\end{cases}
\]

Let $P$ be the orthogonal projection onto $\{\lambda f\}$, the one-dimensional space spanned by $f$, and define $A = 1 - P$. Finally, for $k(x) \in \mathcal{H}$ we define $B$ via

\[(Bk)(x) = \int_{1-x}^1 k(t) \, dt.\]

It is easily checked that $B$ is Hermitian.

Since $Bg = f$, we have $N(AB) = \{\lambda g\}$, and clearly, $N(BA) = \{\lambda f\}$. Thus, both null spaces are one dimensional; we will show that $AB$ and $BA$ are not...
similar by showing $\dim N([AB]^2) = 1$ while $\dim N([BA]^2) = 2$. Notice that $(f,g) = 0$, which implies $Ag = g$, and therefore $ABAg = Af = 0$; hence $g \in N([BA]^2)$. If $\dim N([AB]^2) > 1$, then $g$ belongs to the range of $AB$, and thus $g + \lambda f$ is in the range of $B$ for some scalar $\lambda$. But this is impossible because the range of $B$ consists of absolutely continuous functions. This completes the proof.

3. Fredholm operators. In this section, $\mathcal{R}_X$ will denote the orthogonal projection onto the closure of the range of $X$, and $\mathcal{N}_X$ will be the projection onto the null space. If $P$ is a projection, the symbol $P$ will also be used to denote the subspace $P(\mathbb{C})$; it will be clear from the context which is meant. For the basic facts about Fredholm operators which will be used, the reader is referred to [1]. If $S$ and $T$ are similar, we will write $S \sim T$. Radjavi and Williams [2] have conjectured that every operator which is similar to its adjoint is a product of two Hermitians; if this is true, then whenever $AB \sim BA$, we should be able to find an invertible Hermitian $H$ such that $HAB = BAH$. In all of the following results, the operator which establishes the similarity will be Hermitian. We begin with a simple extension of the finite-dimensional result.

**Proposition 1.** If $A$ and $B$ are Hermitian and $\mathcal{R}_B$ is finite, then $AB \sim BA$.

**Proof.** Let $Q = \sup(\mathcal{R}_B, \mathcal{R}_{AB})$; $Q$ is invariant for both $B$ and $AB$ and thus $\dim N([AB]^2) = \dim N([BA]^2) = 1$. Notice that $(f,g) = 0$, which implies $Ag = g$, and therefore $ABAg = Af = 0$; hence $g \in N([BA]^2)$. If $\dim N([AB]^2) > 1$, then $g$ belongs to the range of $AB$, and thus $g + \lambda f$ is in the range of $B$ for some scalar $\lambda$. But this is impossible because the range of $B$ consists of absolutely continuous functions. This completes the proof.

The main interest of Proposition 1 is that it does not extend to the case when $B$ is compact. In the above example, $A$ is as simple as possible—a projection with one-dimensional null space—and $B$ is compact, and similarity does not hold.

**Lemma 1.** If $E_n$ is $n$-dimensional Euclidean space, $P, Q$ two subspaces of dimension $m$, then there exists an invertible Hermitian $H$ on $E_n$ such that $H$ maps $P$ onto $Q$.

**Proof.** We note that the result is obvious if $P$ and $Q$ are orthogonal; further, we know that $PQ \sim QP$, and therefore $\dim N(PQ) = \dim N(QP)$. But $\dim N(PQ) = \dim (1 - Q) + \dim (Q \cap 1 - P)$ and likewise for $QP$. However, $\dim (1 - Q) = \dim (1 - P) = n - m$, and thus $\dim (Q \cap 1 - P) = \dim (P \cap 1 - Q)$. From our above remark, and since $Q \cap 1 - P$ is orthogonal to $P \cap 1 - Q$, we may assume that $P \cap 1 - Q = Q \cap 1 - P = 0$. But now $\dim \mathcal{R}_{PQ} = m$, and therefore $\mathcal{R}_{PQ} = P, \mathcal{R}_{QP} = Q$. Now $HPQ = QPH$ for an invertible Hermitian $H$, and clearly $H$ satisfies the requirement of the lemma.

**Proposition 2.** If $N_B \cap \mathcal{R}_{AB} = \{0\}$, where $A$ and $B$ are Hermitian and Fredholm, then $AB \sim BA$.  

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Proof. We will show that there exists a mapping $T$, defined on the finite dimensional space $N_{BA}$, which is invertible, Hermitian, and such that $H = \lambda B + TN_{BA}$ is invertible for some $\lambda$. Clearly, for any $H$ of this form, $HAB = BAH$.

Since $B$ is Fredholm and $N_{BA}$ is compact, $H$ is Fredholm for any choice of $T$ and $\lambda \neq 0$; hence, to verify invertibility, it suffices to show $N_H = 0$. Let $x \in N_H$, and write $x = x_r + x_n$, where $x_r \in \mathbb{R}_B$, $x_n \in N_B$. Since $N_B \cap \mathbb{R}_{AB} = \{0\}$, $Bx \neq 0$, and thus $x_r \neq 0$. Now,

$-\lambda Bx_r = TN_{BA} x_r + TN_{BA} x_n$

implies $Bx_r \in N_{BA}$, or $x_r \in \mathbb{R}_B \cap N_{BAB}$. But $N_{BAB} = N_{AB}$, so $x_r \in \mathbb{R}_B \cap N_{AB} = N_{AB} - N_B$. Since $AB$ is Fredholm of index 0, $\dim(N_{AB}) = \dim(N_{BA})$, and thus

(2.1) \[ \dim \langle Bx \rangle | x \in N_{AB} - N_B \rangle \leq \dim(N_{BA}) - \dim(N_B). \]

Also note that $\dim(\mathbb{R}_{N_{BA} N_B}) \leq \dim(N_{BA})$ implies

(2.2) \[ \dim(N_{BA} - \mathbb{R}_{N_{BA} N_B}) \geq \dim(N_{BA}) - \dim(N_B). \]

From (2.1), (2.2), and Lemma 1, we can find an invertible Hermitian $V: N_{BA} \to N_{BA}$ such that $VBx = \mathbb{R}_{N_{BA} N_B}$ for all $x \in N_{AB} - N_B$. Letting $T = V^{-1}$ in (2.0), we get

\[ ||V B x_r|| = ||N_{BA} x_r + N_{BA} x_n|| \leq ||N_{BA} x_r|| \leq ||x_r||. \]

However, $B$ has closed range, which means that $||By|| = \epsilon ||y||$ for all $y \in \mathbb{R}_B$ and some $\epsilon > 0$. Similarly, $||Vy|| \geq \delta ||y||$ for $y \in N_{BA}$ by the invertibility of $V$. Hence,

\[ \lambda \epsilon \delta ||x_r|| \leq \lambda ||V B x_r|| \leq ||x_r|| \]

which is clearly violated if $\lambda > 1/\epsilon \delta$. Thus, $H$ is invertible, and the proof is complete. \[ \square \]

Corollary 1. If $A$ and $B$ are Hermitian, Fredholm, and $A \geq 0$, then $AB \sim BA$.

Proof. Let $A_0 = (A + N_A)^{-1/2}$, which exists because $A + N_A$ is invertible and positive. If $B_1 = A_0^{-1} BA_0^{-1}$, then

$A_0 A B A_0^{-1} = (A_0 A A_0)(A_0^{-1} B A_0^{-1}) = A_0 A B_1 = \mathbb{R}_A B_1.$

We may therefore assume that $A$ is a projection. However, $N_{X \times X} = N_X$ for all operators, so $N_{AB} = N_{BAB} = N_{BAB}$ which says that $N_B \cap \mathbb{R}_{AB} = \{0\}$, and thus Proposition 2 applies. \[ \square \]

The above corollary may also be deduced from Proposition 3, the proof of which requires the following lemma. This lemma was demonstrated by Radjavi and Williams [2] in the course of proving their Theorem 5, so the proof will be omitted.

Lemma 2. If $P$ and $Q$ are linearly independent invariant subspaces of $X$ which
span \( \mathcal{V} \), then there exists an invertible operator \( R \) such that

(a) \( X_1 = R^{-1} X R \) commutes with \( P \),
(b) \( Rp = p \) for \( p \in P \),
(c) \( \sigma(X|_P) = \sigma(X_1|_P) \) where \( \sigma \) is the spectrum and \( X|_P \) is \( X \) considered as an operator from \( P \) into \( P \).

An operator \( X \) is said to have finite descent \( n \) if \( n \) is the smallest integer for which \( X^n \) and \( X^{n+1} \) have the same range. For a Fredholm operator, finite descent is equivalent to zero being an isolated point of the spectrum.

**Proposition 3.** If \( A \) and \( B \) are Hermitian and Fredholm, and if \( AB \) has finite descent \( n \), then \( AB \sim BA \).

**Proof.** Since \( AB \) has index 0, \( N([AB]^n) = N([AB]^{n+1}) \); let \( N_0 = N([AB]^n) \), \( R_0 = \mathcal{R}([AB]^n) \). Theorem 5.41-G of [3] applies, and so \( \{N_0, R_0\} \) satisfies the conditions of Lemma 2, with \( AB \) playing the role of \( X \). Note that \( N_0 \) is finite, and that \( \sigma(AB|_{N_0}) = 0 \). Since \( R^{-1}ABR = (R^{-1}AR^{-1}*)(R^*BR) \) is also a product of Hermitians, Lemma 2 says (with \( N_0 = P \) that we may assume that \( AB \) commutes with \( N_0 \); i.e., \( AB = (\psi \ 0) \) where \( \mathcal{V} = N_0 \oplus N_0^\perp \). By (b) of Lemma 2, \( N_U = 0 \), and since \( U \) is Fredholm with index 0, \( U \) is invertible. We also have \( V^n = 0 \). Now \( B(AB)^n \) is Hermitian, and \( N_{B(AB)^n} = N_{(AB)^n} = N_0 \), and thus \( \mathcal{R}_{B(AB)^n} = N_0^\perp = \mathcal{R}_{(AB)^n} \). Therefore, \( N_0^\perp \) is a reducing subspace for \( B \) on which \( B \) is invertible. Let \( B_1 = B|_{N_0} \). Furthermore, by Theorem 1 of [2] and the fact that \( \sigma(V) = 0 \), we can find an invertible Hermitian \( H \) defined on \( N_0 \) such that \( HV = V^*H \). Hence,

\[
\begin{pmatrix}
H & 0 \\
0 & B_1
\end{pmatrix}
\begin{pmatrix}
V & 0 \\
0 & U
\end{pmatrix} =
\begin{pmatrix}
V^* & 0 \\
0 & U^*
\end{pmatrix}
\begin{pmatrix}
H & 0 \\
0 & B_1
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
H & 0 \\
0 & B_1
\end{pmatrix}
\]

is invertible. \( \square \)

**References**


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