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LIMITS OF SUCCESSIVE CONVOLUTIONS

JOHN C. MARTIN

ABSTRACT. On an arbitrary compact, zero-dimensional, Abelian group, if $\mu_0, \mu_1, \ldots$ is a sequence of probability measures, a condition on these measures is given which is necessary and sufficient for each of the sequences $\mu_t, \mu_t * \mu_{t+1}, \mu_t * \mu_{t+1} * \mu_{t+2}, \ldots$ of successive convolutions to converge to Haar measure in the weak-star topology. Some simple consequences of the theorem are noted.

Let G be a compact, Abelian group and μ a Borel probability measure on G. If we denote by $\mu^{(n)}$ the nth convolution power of μ, the following result is well known.

THEOREM [1, THEOREM 8]. *$\mu^{(n)}$ converges to the normalized Haar measure λ_G in the weak-star topology if and only if the support of μ is not contained in a coset of any proper closed subgroup of G.*

In this paper we shall be concerned primarily with zero-dimensional, compact, Abelian groups. For such a group G we consider, in place of $\mu^{(n)}$, successive convolutions of the form $\mu_k * \mu_{k+1} * \cdots * \mu_{k+m}$ (henceforth abbreviated $\mu_{k,m}$), where $\mu_1, \mu_2, \ldots$ is any sequence of Borel probability measures on G; we derive a necessary and sufficient condition for $\mu_{k,m}$ to converge to λ_G in the weak-star topology for every k.

For background material, including a more complete discussion of convolutions, the reader is referred to [3].

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If G is any compact, Abelian group, $C(G)$ will denote the Banach space of continuous complex-valued functions on G and $M(G) \subseteq C^*(G)$ the set of Borel probability measures on G. For $\mu, \nu \in M(G)$, the convolution $\mu * \nu$ is defined by

$$\int_G f(x)\, d(\mu * \nu)(x) = \int_G \int_G f(x+y)\, d\mu(x)\, d\nu(y) \qquad (f \in C(G)).$$

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Now if H is a subgroup of G, H is open and closed if and only if the index $(G:H)$ of H in G is finite; if G is zero-dimensional, this will be true whenever H is the kernel of a continuous character on G. Let H be an open-closed subgroup, and let $\mu \in M(G)$. We define $\mu^H \in M(G/H)$ as follows:

$$\mu^H\{[x]\} = \mu([x]) \qquad ([x] \in G/H).$$

Then it may be verified that for $\mu, \nu \in M(G)$,

$$(\mu * \nu)^H = \mu^H * \nu^H. \tag{1}$$

We denote by $m_H(\mu)$ the maximum of the numbers $\mu^H\{[x]\}$.

THEOREM. *Let G be a compact, zero-dimensional, Abelian group, and let $\mu_1, \mu_2, \ldots \in M(G)$. Then $\lim_{m\to\infty} \mu_{k,m} = \lambda_G$ in the weak-star topology for every k if and only if*

(2) *for every proper open-closed subgroup H of G, $\prod_{t=1}^{\infty} m_H(\mu_t) = 0$.*

REMARK. This result, stated in a different form and without reference to convolutions, is obtained in Lemma 3 of [2] in the case where G is the group of integers modulo two. In this case there is a simple proof involving the formula

$$|\mu_{k,m}\{0\} - \mu_{k,m}\{1\}| = \prod_{t=k}^{k+m} |\mu_t\{0\} - \mu_t\{1\}|.$$

Our proof uses two preparatory lemmas.

LEMMA 1. *For any compact, Abelian group G and any sequence $\mu_1, \mu_2, \ldots \in M(G)$, $\mu_{k,m}$ converges to λ_G in the weak-star topology for every k if and only if $\prod_{t=k}^{\infty} |\hat{\mu}_t(\gamma)| = 0$ for each k and each nontrivial character γ on G.*

PROOF. It is well known that a sequence ν_n in $M(G)$ converges to $\nu \in M(G)$ in the weak-star topology if and only if $\hat{\nu}_n$ converges pointwise to $\hat{\nu}$. We have $\hat{\lambda}_G(1) = 1$, $\hat{\mu}_t(1) = 1$ for each t, and $\hat{\lambda}_G(\gamma) = 0$ for $\gamma \neq 1$. The lemma follows from the fact that $\hat{\mu}_{k,m} = \hat{\mu}_k \hat{\mu}_{k+1} \cdots \hat{\mu}_{k+m}$. Q.E.D.

LEMMA 2. *Suppose $r_i \geqslant 0$ $(i = 0, 1, \ldots, j-1)$ and $\sum_{i=0}^{j-1} r_i = 1$. Let $R = \max r_i$, and denote by w the jth root of unity $e(1/j) = \exp(2\pi i/j)$. Then*

$$\left|\sum_{i=0}^{j-1} r_i w^i\right| \leqslant |R + (1-R)w|.$$

The proof is straightforward.

PROOF OF THEOREM. Suppose (2) holds. (In fact, we need only assume that it holds for subgroups H which are kernels of continuous characters on G.) Let γ be any continuous character other than 1; let $\gamma(G) = \{1, w, \ldots, w^{j-1}\}$ (where $w = e(1/j)$), and let $H = \operatorname{Ker} \gamma$. Then for $t \geqslant 1$,

$$\hat{\mu}_t(\gamma) = \int_G \overline{\gamma(x)}\, d\mu_t(x) = \sum_{i=0}^{j-1} r_{t,i} w^i,$$

where $r_{t,i} = \mu_t(\gamma^{-1}(w^{-i}))$. Let $R_t = \max_i r_{t,i} = m_H(\mu_t)$. Then from Lemma 2,

$$|\hat{\mu}_t(\gamma)| \leqslant |R_t + (1 - R_t)w| \leqslant (1 - 2R_t(1 - R_t)(1 - \cos(2\pi/j)))^{1/2}.$$

By Lemma 1, it is sufficient to show that

$$\prod_{t=k}^{\infty} (1 - 2R_t(1 - R_t)(1 - \cos(2\pi/j))) = 0 \qquad (k > 0),$$

which will be true if $\sum_{t=1}^{\infty} R_t(1 - R_t)$ diverges. But since $R_t \geqslant 1/j$, the conclusion follows from (2).

The proof in the other direction holds for any compact Abelian group and any closed subgroup H and is a straightforward application of (1). Q.E.D.

A few observations on the theorem are in order. First, it is easy to see that in the special case where $\mu_1 = \mu_2 = \cdots = \mu$, condition (2) reduces to the statement in the theorem of Kawada and Itô at the beginning of the paper.

We consider next the special case in which

$$d\mu_t = f_t\, d\lambda_G,$$

where $f_t \in L^2(G, \lambda_G)$, $f_t \geqslant 0$, and $\int_G f_t\, d\lambda_G = 1$. Then since $f_t \in L^2$, each of the sequences $f_k * f_{k+1}, f_k * f_{k+1} * f_{k+2}, \ldots$ is equicontinuous and therefore has a uniformly convergent subsequence. It follows that each of these sequences converges uniformly to 1 if and only if, for every proper open-closed subgroup H, $\prod_{t=1}^{\infty} m_H(f_t) = 0$ (where $m_H(f_t) = m_H(\mu_t)$).

Finally, we note that if (2) holds for any compact Abelian group G, then $\mu_{k,m}(f)$ converges to $\lambda_G(f)$ for each k, provided that f is a continuous function which is constant on cosets of some open-closed subgroup; zero-dimensionality assures that all trigonometric polynomials have this property.

One can consider the same question if the group G is non-Abelian. In this case, the statement of the theorem will require at least some modification. Clearly, a necessary condition for $\mu_{k,m}$ to converge to λ_G for each k is that $\prod_{t=1}^{\infty} \mu_t(H) = 0$ for every proper open-closed subgroup H; hence, the condition that (2) hold for every proper, normal, open-closed subgroup is not sufficient (though it is necessary). On the other hand, condition (2) as stated is not in general necessary: if G is the permutation group S_3, $H = \{e, \phi\}$ is a subgroup of order 2, $\psi \notin H$, and $\mu\{\psi\} = \mu\{\psi\phi\} = \frac{1}{2}$, we see that $\mu^{(n)}$ converges to λ_G although the measure of the left coset ψH is 1.

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Department of Mathematics, North Dakota State University, Fargo, North Dakota 58102