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FIXED POINTS AND ITERATION OF A NONEXPANSIVE MAPPING IN A BANACH SPACE

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ABSTRACT. The following result is shown. If T is a nonexpansive mapping from a closed convex subset D of a Banach space into a compact subset of D and x_1 is any point in D, then the sequence $\{x_n\}$ defined by $x_{n+1} = 2^{-1}(x_n + Tx_n)$ converges to a fixed point of T. As a matter of fact, a theorem which includes this result is proved. Furthermore, a similar result is obtained under certain restrictions which do not imply the assumption on the compactness of T.

Throughout this paper we consider the following iterative procedure, which is a special case of the generalized iteration method introduced by W. R. Mann [7].

DEFINITION. If D is a subset of a Banach space X, T is a mapping from D into X, and $x_1 \in D$, then $M(x_1, t_n, T)$ is the sequence $\{x_n\}_{n=1}^{\infty}$ defined by $x_{n+1} = (1 - t_n)x_n + t_n Tx_n$, where $\{t_n\}_{n=1}^{\infty}$ is a real sequence. If a point x_1 and a sequence $\{t_n\}_{n=1}^{\infty}$ satisfy the following three conditions:

$$\sum_{n=1}^{\infty} t_n = \infty, \tag{1}$$

$$0 \leqslant t_n \leqslant b < 1 \quad \text{for all positive integers } n, \tag{2}$$

and

$$x_n \in D \quad \text{for all positive integers } n,$$

then x_1 and $\{t_n\}_{n=1}^{\infty}$ will be said to satisfy Condition A.

Note that if $t_n \in [a, b]$ for all positive integers n and $0 < a \leqslant b < 1$, then it is obvious that the sequence $\{t_n\}_{n=1}^{\infty}$ satisfies (1) and (2).

These iteration methods have been investigated by Krasnosel'skiĭ [6], Edelstein [3], Outlaw [9], Dotson [2] and others. They showed that these iterative methods may be used to find a fixed point of a nonexpansive mapping T mainly in a uniformly convex Banach space or a strictly convex Banach space, where a mapping T from a subset D of a Banach space X into X is called a nonexpansive mapping if T satisfies the condition that $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in D$.

In this paper we study the iterative method for nonexpansive mappings

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without any assumption on convexity of the Banach space.

LEMMA 1. *Let $\{s_i\}_{i=1}^{\infty}$ be a sequence in the real numbers and let $\{u_i\}_{i=1}^{\infty}$ be a sequence in a Banach space X. Then for any positive integer N,*

$$\begin{aligned}(3)\qquad &\left(\prod_{i=1}^{N-1} s_i\right)\left(\sum_{i=1}^{N}(1-s_i)u_i\right)\\ &= \left(1-\prod_{i=1}^{N} s_i\right)u_N - \sum_{i=1}^{N-1}\left\{\left(\prod_{j=i+1}^{N-1} s_j\right)\left(1-\prod_{j=1}^{i} s_j\right)(u_{i+1}-s_iu_i)\right\}.\end{aligned}$$

If X is the real line and $u_i = 1$ for all i, we have the special case

$$\begin{aligned}(4)\qquad &\left(\prod_{i=1}^{N-1} s_i\right)\left(\sum_{i=1}^{N}(1-s_i)\right)\\ &= 1-\prod_{i=1}^{N} s_i - \sum_{i=1}^{N-1}\left\{\left(\prod_{j=i+1}^{N-1} s_j\right)\left(1-\prod_{j=1}^{i} s_j\right)(1-s_i)\right\}.\end{aligned}$$

Here and hereafter we agree that $\sum_{i=m}^{n}$ and $\prod_{i=m}^{n}$ are defined to be 0 and 1, respectively, for $n < m$.

PROOF. When $N = 1$, the result is trivial. Supposing that (3) is true for some $N \geqslant 1$, we have

$$\begin{aligned}&\sum_{i=1}^{N}\left\{\left(\prod_{j=i+1}^{N} s_j\right)\left(1-\prod_{j=1}^{i} s_j\right)(u_{i+1}-s_iu_i)\right\}\\
&\qquad = s_N\sum_{i=1}^{N-1}\left\{\left(\prod_{j=i+1}^{N-1} s_j\right)\left(1-\prod_{j=1}^{i} s_j\right)(u_{i+1}-s_iu_i)\right\}\\
&\qquad\quad -s_N\left(1-\prod_{i=1}^{N} s_i\right)u_N + \left(1-\prod_{i=1}^{N} s_i\right)u_{N+1}\\
&\qquad = s_N\left\{\left(1-\prod_{i=1}^{N} s_i\right)u_N - \left(\prod_{i=1}^{N-1} s_i\right)\left(\sum_{i=1}^{N}(1-s_i)u_i\right)\right\}\\
&\qquad\quad -s_N\left(1-\prod_{i=1}^{N} s_i\right)u_N + \left(1-\prod_{i=1}^{N} s_i\right)u_{N+1}\\
&\qquad = -\left(\prod_{i=1}^{N} s_i\right)\left(\sum_{i=1}^{N}(1-s_i)u_i\right) + \left(1-\prod_{i=1}^{N} s_i\right)u_{N+1},\end{aligned}$$

from which it follows that

{the right-hand side of (3) with $N+1$ for N}

$$\begin{aligned}&= \left(1-\prod_{i=1}^{N+1} s_i\right)u_{N+1} - \sum_{i=1}^{N}\left\{\left(\prod_{j=i+1}^{N} s_j\right)\left(1-\prod_{j=1}^{i} s_j\right)(u_{i+1}-s_iu_i)\right\}\\
&= \left(1-s_{N+1}\prod_{i=1}^{N} s_i\right)u_{N+1} + \left(\prod_{i=1}^{N} s_i\right)\left(\sum_{i=1}^{N}(1-s_i)u_i\right) - \left(1-\prod_{i=1}^{N} s_i\right)u_{N+1}\\
&= \left(\prod_{i=1}^{N} s_i\right)\left(\sum_{i=1}^{N+1}(1-s_i)u_i\right).\end{aligned}$$

By induction this completes the proof.

LEMMA 2. *Let D be a subset of a Banach space X and let T be a nonexpansive mapping from D into X. If there exist x_1 and $\{t_n\}_{n=1}^{\infty}$ that satisfy Condition* A *and $M(x_1, t_n, T)$ is bounded, then $x_n - Tx_n$ converges to zero as $n \to \infty$.*

PROOF. Since T is a nonexpansive mapping, we have

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &= \|(1 - t_n)x_n + t_n Tx_n - Tx_{n+1}\| \\
&= \|(1 - t_n)(x_n - Tx_n) + Tx_n - Tx_{n+1}\| \\
&\leqslant (1 - t_n)\|x_n - Tx_n\| + \|x_n - x_{n+1}\| \\
&= (1 - t_n)\|x_n - Tx_n\| + \|x_n - ((1 - t_n)x_n + t_n Tx_n)\| \\
&= \|x_n - Tx_n\|.
\end{aligned}$$

Thus the sequence $\{\|x_n - Tx_n\|\}_{n=1}^{\infty}$ is nonincreasing and bounded below, so $\lim_{n\to\infty}\|x_n - Tx_n\|$ exists.

Suppose that $\lim_{n\to\infty}\|x_n - Tx_n\| = r > 0$. That is, for any $\varepsilon > 0$, there exists an integer m such that

$$r \leqslant \|x_{m+i} - Tx_{m+i}\| \leqslant (1 + \varepsilon)r \quad \text{for all positive integers } i. \tag{5}$$

Then since T is nonexpansive,

$$\begin{aligned}
&\|(Tx_{m+i+1} - x_{m+i+1}) - (1 - t_{m+i})(Tx_{m+i} - x_{m+i})\| \\
&\quad = \|T((1 - t_{m+i})x_{m+i} + t_{m+i}Tx_{m+i}) \\
&\qquad\quad -((1 - t_{m+i})x_{m+i} + t_{m+i}Tx_{m+i}) - (1 - t_{m+i})(Tx_{m+i} - x_{m+i})\| \\
&\quad = \|T((1 - t_{m+i})x_{m+i} + t_{m+i}Tx_{m+i}) - Tx_{m+i}\| \\
&\quad \leqslant t_{m+i}\|x_{m+i} - Tx_{m+i}\| \leqslant t_{m+i}(1 + \varepsilon)r.
\end{aligned} \tag{6}$$

Since $\{x_n\}_{n=1}^{\infty}$ is bounded and $\{t_n\}_{n=1}^{\infty}$ satisfies condition (1), there exists an integer N such that

$$r\sum_{i=1}^{N-1} t_{m+i} \leqslant \delta(M) + 1 \leqslant r\sum_{i=1}^{N} t_{m+i} \tag{7}$$

where $\delta(M)$ is defined by $\sup\{\|x_i - x_j\|; 0 < i,j < \infty\}$.

Now setting $s_i = 1 - t_{m+i}$ and $u_i = Tx_{m+i} - x_{m+i}$ for all positive integers i, we get from (6),

$$\begin{aligned}
\|u_{i+1} - s_i u_i\| &= \|Tx_{m+i+1} - x_{m+i+1} - (1 - t_{m+i})(Tx_{m+i} - x_{m+i})\| \\
&\leqslant t_{m+i}(1 + \varepsilon)r = (1 - s_i)(1 + \varepsilon)r
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
x_{m+N+1} - x_{m+1} &= \sum_{i=1}^{N} \{((1 - t_{m+i})x_{m+i} + t_{m+i}Tx_{m+i}) - x_{m+i}\} \\
&= \sum_{i=1}^{N} t_{m+i}(Tx_{m+i} - x_{m+i}) = \sum_{i=1}^{N} (1 - s_i)u_i.
\end{aligned} \tag{9}$$

Thus using Lemma 1, we have from (9), (3), (5) and (8) that

$$\begin{aligned}
\left(\prod_{i=1}^{N-1} s_i\right)\|x_{m+N+1}-x_{m+1}\| &= \left\|\left(\prod_{i=1}^{N-1} s_i\right)\left(\sum_{i=1}^{N}(1-s_i)u_i\right)\right\| \\
&\geqslant \left(1-\prod_{i=1}^{N} s_i\right)\|u_N\| - \sum_{i=1}^{N-1}\left\{\left(\prod_{j=i+1}^{N-1} s_j\right)\left(1-\prod_{j=1}^{i} s_j\right)\|u_{i+1}-s_iu_i\|\right\} \\
&\geqslant \left(1-\prod_{i=1}^{N} s_i\right)r - \sum_{i=1}^{N-1}\left\{\left(\prod_{j=i+1}^{N-1} s_j\right)\left(1-\prod_{j=1}^{i} s_j\right)(1-s_i)(1+\varepsilon)r\right\} \\
&= \left[1-\prod_{i=1}^{N} s_i - \sum_{i=1}^{N-1}\left\{\left(\prod_{j=i+1}^{N-1} s_j\right)\left(1-\prod_{j=1}^{i} s_j\right)(1-s_i)\right\}\right]r \\
&\quad -\varepsilon r\sum_{i=1}^{N-1}\left\{\left(\prod_{j=i+1}^{N-1} s_j\right)\left(1-\prod_{j=1}^{i} s_j\right)(1-s_i)\right\},
\end{aligned}$$

since $s_i = 1 - t_{m+i} \geqslant 1 - b > 0$, which implies from (4) and (7) that

$$\begin{aligned}
\|x_{m+N+1}-x_{m+1}\| &\geqslant r\sum_{i=1}^{N}(1-s_i) - \varepsilon r\left(\prod_{i=1}^{N-1} s_i\right)^{-1} \\
&\qquad \times\left\{1-\prod_{i=1}^{N} s_i - \left(\prod_{i=1}^{N-1} s_i\right)\left(\sum_{i=1}^{N}(1-s_i)\right)\right\} \\
&\geqslant r\sum_{i=1}^{N}(1-s_i) - \varepsilon r\left(\prod_{i=1}^{N-1} s_i\right)^{-1} \qquad (10)\\
&= r\sum_{i=1}^{N} t_{m+i} - \varepsilon r\prod_{i=1}^{N-1}(1-t_{m+i})^{-1} \\
&\geqslant \delta(M) + 1 - \varepsilon r\prod_{i=1}^{N-1}(1-t_{m+i})^{-1}.
\end{aligned}$$

Since $\log(1+y) \leqslant y$ for any $y \in (-1, \infty)$, we have from (2) and (7),

$$\begin{aligned}
\prod_{i=1}^{N-1}(1-t_{m+i})^{-1} &= \prod_{i=1}^{N-1}\left(1+t_{m+i}(1-t_{m+i})^{-1}\right) \\
&= \exp\left\{\sum_{i=1}^{N-1}\log\left(1+t_{m+i}(1-t_{m+i})^{-1}\right)\right\} \\
&\leqslant \exp\left\{\sum_{i=1}^{N-1} t_{m+i}(1-t_{m+i})^{-1}\right\} \\
&\leqslant \exp\left\{(1-b)^{-1}\sum_{i=1}^{N-1} t_{m+i}\right\} \\
&\leqslant \exp\{(1-b)^{-1}(\delta(M)+1)r^{-1}\}.
\end{aligned}$$

From this and (10) we get that

$$\begin{aligned}
\delta(M) + 1 - \varepsilon r\exp\{(1-b)^{-1}(\delta(M)+1)r^{-1}\} \\
\leqslant \|x_{m+N+1}-x_{m+1}\| \leqslant \delta(M).
\end{aligned}$$

Since ε is an arbitrary positive number, it follows that $\delta(M) + 1 \leqslant \delta(M)$. This contradiction completes the proof.

REMARK. Let T be a nonexpansive mapping from a convex set D in a Banach space into a bounded subset of D and let $(1-t)I + tT$ be denoted by T_t, where I is an identity map and $0 < t < 1$. Then $M(x_1, t, T)$ is bounded since it is a sequence in the convex hull of the union of $T(D)$ and the point x_1. Also it is clear that $T_t^n x_1 - T_t^{n-1} x_1 = t(Tx_n - x_n)$. Therefore we have by Lemma 2 that T_t is asymptotically regular (i.e. for any $x \in D$, $\|T_t^{n+1}x - T_t^n x\| \to 0$ as $n \to \infty$).

Fixed points and iterative process for compact mappings. Now we shall prove a fixed point theorem for a nonexpansive compact mapping and show that the iterative process $M(x_1, t_n, T)$ may be used to find the fixed point.

THEOREM 1. *Let D be a closed subset of a Banach space X and let T be a nonexpansive mapping from D into a compact subset of X. If there exist x_1 and $\{t_n\}_{n=1}^{\infty}$ that satisfy Condition* A, *then T has a fixed point in D and $M(x_1, t_n, T)$ converges to a fixed point of T.*

PROOF. Let D_0 denote the closure of the convex hull of the union of $T(D)$ and the point x_1. A well-known theorem of Mazur implies that D_0 is compact. The sequence $M(x_1, t_n, T)$ clearly belongs to D_0. From this and Condition A, it immediately follows that $\{x_n\}_{n=1}^{\infty}$ is a compact sequence in D. Hence there is a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ that converges to a point u, which obviously belongs to D since D is closed. And it is clear that $\lim_{i\to\infty} \|Tx_{n_i} - x_{n_i}\| = 0$ since Lemma 2 is applicable from the boundedness of D_0.

Now since T is nonexpansive,

$$\begin{aligned} \|Tu - u\| &= \|Tu - Tx_{n_i} + Tx_{n_i} - x_{n_i} + x_{n_i} - u\| \\ &\leqslant 2\|u - x_{n_i}\| + \|Tx_{n_i} - x_{n_i}\|, \end{aligned}$$

which implies that u is a fixed point of T since $\lim_{i\to\infty} \|u - x_{n_i}\| = 0$ and $\lim_{i\to\infty} \|Tx_{n_i} - x_{n_i}\| = 0$.

Further,

$$\begin{aligned} \|x_{n+1} - u\| &= \|(1-t_n)x_n + t_n Tx_n - u\| \\ &= \|(1-t_n)(x_n - u) + t_n(Tx_n - Tu)\| \leqslant \|x_n - u\| \end{aligned} \tag{11}$$

for any positive integer n. For any $\varepsilon > 0$ there exists an integer n_0 such that $\|x_{n_0} - u\| < \varepsilon$, so we obtain from (11) that $\|x_n - u\| < \varepsilon$ for any integer $n \geqslant n_0$. Therefore $M(x_1, t_n, T)$ converges to u, a fixed point of T.

As an immediate consequence of Theorem 1, we have the following corollaries.

COROLLARY 1. *Let D be a closed subset of a Banach space X and let T be a nonexpansive mapping from D into a compact subset of X. If there exists $t \in (0, 1)$ such that $(1-t)x + tTx \in D$ for all $x \in D$, then T has a fixed point in D and for any $x_1 \in D$, $M(x_1, t, T)$ converges to a fixed point of T.*

COROLLARY 2. *Let D be a closed convex subset of a Banach space X and let T*

be a nonexpansive mapping from D into a compact subset of D. Then T has a fixed point in D and $M(x_1, 2^{-1}, T)$ *converges to a fixed point of T for any* $x_1 \in D$.

Note that the first part of Corollary 2 is a special case of a fixed point theorem of Schauder.

Corollary 2 was proved for uniformly convex spaces by Krasnosel'skiĭ [6] and strictly convex spaces by Edelstein [3].

Fixed points and iterative process for noncompact mappings. Next we shall consider the iterative process for a nonexpansive mapping without the assumption on the compactness of T.

Let D be a subset of a Banach space X. A mapping $T: D \to X$ with a nonempty fixed points set F in D will be said to satisfy Condition B if there is a nondecreasing function $f: [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0, \infty)$, such that $\|x - Tx\| \geqslant f(d(x, F))$ for all $x \in D$, where $d(x, F) = \inf\{\|x - z\|; z \in F\}$. This condition is due to Senter and Dotson [10].

THEOREM 2. *Let D be a closed subset of a Banach space X and let* $T: D \to X$ *be a nonexpansive mapping with a nonempty fixed points set F in D. If T satisfies Condition* B *and there exist* x_1 *and* $\{t_n\}_{n=1}^{\infty}$ *that satisfy Condition* A, *then* $M(x_1, t_n, T)$ *converges to a member of F.*

PROOF. The theorem is trivial if $x_1 \in F$, so we assume $x_1 \in D - F$. For any $u \in F$ we have that $\|Tx_n - u\| \leqslant \|x_n - u\|$ and so we get that

$$\|x_{n+1} - u\| = \|(1 - t_n)x_n + t_n Tx_n - u\| \leqslant \|x_n - u\| \tag{12}$$

which implies that $d(x_{n+1}, F) \leqslant d(x_n, F)$ for all positive integers n. The sequence $\{d(x_n, F)\}_{n=1}^{\infty}$ is nonincreasing and bounded below, so there exists $\lim_{n\to\infty} d(x_n, F)$, which we denote by r.

By the definition of f, we have

$$\|x_n - Tx_n\| \geqslant f(d(x_n, F)) \geqslant f(r). \tag{13}$$

Since it follows from (12) that $M(x_1, t_n, T)$ is a bounded sequence in D, we have from Lemma 2 and (13) that $f(r) = 0$. Hence we get that

$$\lim_{n\to\infty} d(x_n, F) = r = 0.$$

Now we shall show that $M(x_1, t_n, T)$ converges to a member of F. Since $\lim_{n\to\infty} d(x_n, F) = 0$, for any positive integer i there exist $N_i > 0$ and $u_i \in F$ such that $\|x_{N_i} - u_i\| < 2^{-i}$, which implies from (12) that $\|x_n - u_i\| < 2^{-i}$ for all $n \geqslant N_i$. We require $N_{i+1} \geqslant N_i$ for all $i > 0$. Then we have that for any integers i and j such that $i < j$,

$$\begin{aligned}\|u_i - u_j\| &\leqslant \|u_i - x_{N_{i+1}}\| + \|x_{N_{i+1}} - u_{i+1}\| + \|u_{i+1} - x_{N_{i+2}}\| \\ &\quad + \cdots + \|u_{j-1} - x_{N_j}\| + \|x_{N_j} - u_j\| \\ &\leqslant 2^{-i} + 2^{-i-1} + 2^{-i-1} + 2^{-i-2} + \cdots + 2^{-j+1} + 2^{-j} \\ &= 3(2^{-i} - 2^{-j})\end{aligned}$$

which implies $\{u_i\}_{i=1}^{\infty}$ is a Cauchy sequence, so there exists v such that $v = \lim_{i\to\infty} u_i$ and v belongs to F since F is closed. For any $\varepsilon > 0$ there exists $i_0 > 0$ such that $2^{-i_0} < 2^{-1}\varepsilon$ and $\|u_{i_0} - v\| < 2^{-1}\varepsilon$, so we have that

$$\|x_n - v\| \leqslant \|x_n - u_{i_0}\| + \|u_{i_0} - v\| \leqslant 2^{-i_0} + \|u_{i_0} - v\| < \varepsilon \quad \text{for all } n > N_{i_0}.$$

Therefore $M(x_1, t_n, T)$ converges to the point v of F.

COROLLARY 3. *Let D be a closed convex subset of X and let T: $D \to D$ be a nonexpansive mapping with a nonempty fixed points set F. If T satisfies Condition* B, *then for any $x_1 \in D$ and any $\{t_n\}_{n=1}^{\infty}$ satisfying* (1) *and* (2), $M(x_1, t_n, T)$ *converges to a member of F.*

If X is a uniformly convex Banach space and $0 < a \leqslant t_n \leqslant b < 1$ for all integers $n > 0$, the analog of this corollary was obtained by Senter and Dotson [10].

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REFERENCES

1. G. Darbo, *Punti uniti in transformazioni a condominio non compatto*, Rend. Sem. Mat. Univ. Padova **24** (1955), 84–92. MR **16**, 1140.

2. W. G. Dotson, Jr., *On the Mann iterative process*, Trans. Amer. Math. Soc. **149** (1970), 65–73. MR **41** #2477.

3. M. Edelstein, *A remark on a theorem of M. A. Krasnoselski*, Amer. Math. Monthly **73** (1966), 509–510. MR **33** #3072.

4. G. W. Groetsch, *A note on segmenting Mann iterates*, J. Math. Anal. Appl. **40** (1972), 369–372. MR **49** #5954.

5. S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147–150. MR **49** #1243.

6. M. A. Krasnosel'skiĭ, *Two remarks on the method of successive approximations*, Uspehi Mat. Nauk **10** (1955), no. 1 (63), 123–127. (Russian) MR **16**, 833.

7. W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510. MR **14**, 988.

8. R. D. Nussbaum, *The fixed point index for local condensing maps*, Ann. Mat. Pura. Appl. (4) **89** (1971), 213–258. MR **47** #903.

9. C. L. Outlaw, *Mean value iteration of nonexpansive mappings in a Banach space*, Pacific J. Math. **30** (1969), 747–750. MR **40** #807.

10. H. F. Senter and W. G. Dotson, *Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc. **44** (1974), 375–380. MR **49** #11333.

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